BIANCHI'S BÄCKLUND TRANSFORMATION FOR HIGHER DIMENSIONAL QUADRICS

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Abstract. We provide a generalization of Bianchi's Bäcklund transformation from 2-dimensional quadrics to higher dimensional quadrics. The starting point of our investigation is the higher dimensional (infinitesimal) version of Bianchi's main four theorems on the theory of deformations of quadrics and Bianchi's treatment of the Bäcklund transformation for diagonal paraboloids via conjugate systems.

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1. Introduction

In 1859 the French Academy posed the problem:

To find all surfaces applicable to a given one.

It became the driving force which led to the flourishing of the classical differential geometry in the second half of the XIX$^{\text{th}}$ century and its profound study by illustrious geometers led to interesting results (see Bianchi [4],[5],[6], Darboux [10], Eisenhart [11],[12], Sabitov [14] and its references for results up to 1990's or Spivak ([15], Vol **5**)). Today it is still an open problem in its full generality, but basic familiar results like the Gauß-Bonnet Theorem and the Codazzi-Mainardi equations (independently discovered also by Peterson) were first communicated to the French Academy. A list (most likely incomplete) of the winners of the prize includes Bianchi, Bonnet, Guichard, Weingarten.

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Up to 1899 deformations of the (pseudo-)sphere and isotropic quadrics without center (from a metric point of view they can be considered as metrically degenerate quadrics without center) together with their *Bäcklund* (B) transformation and the complementary transformation of deformations of surfaces of revolution were investigated by geometers such as Bäcklund, Bianchi, Darboux, Goursat, Hazzidakis, Lie, Weingarten, etc.

In 1899 Guichard discovered that when quadrics with(out) center and of revolution around the focal axis roll on their deformations their foci describe constant mean curvature (minimal) surfaces (and Bianchi proved the converse: all constant mean curvature (minimal) surfaces can be realized in this way).

With Guichard's result the race to find the deformations of general quadrics was on; it ended with Bianchi's discovery [3] from 1906 of the B transformation for quadrics and the *applicability correspondence provided by the Ivory affinity* (ACPIA).

Note also that Peterson's work on deformations of general quadrics preceded that of Bianchi, Calapso, Darboux, Guichard and Ţiţeica's from the years 1899-1906 by two decades, but unfortunately most of his works (including his independent discovery of the Codazzi-Mainardi equations and of the Gauß-Bonnet Theorem) were made known to Western Europe mainly after they were translated in 1905 from Russian to French (as is the case with his deformations of quadrics [13], originally published in 1883 in Russian).

The work of these illustrious geometers on deformations in $\mathbb{C}^3$ of quadrics in $\mathbb{C}^3$ (there is no other class of surfaces for which an interesting theory of deformation has been built) is one of the crowning achievements of the golden age of classical geometry of surfaces (Darboux split the prize of the French Academy between Bianchi and Guichard for solving the problem for quadrics (Guichard had other results including the G transformation, but the G transformation turned out to be the composition of two B transformations); it was solved in the sense that solutions depending on arbitrarily many constants are produced by algebraic procedures once an 1-dimensional family of Riccati equations is presumed solved) and at the same time it opened new areas of research (such as affine and projective differential geometry) continued later by other illustrious geometers (Blaschke, Cartan, etc).

Note that Calapso in [8] has put Bianchi's B transformation of deformations in $\mathbb{C}^3$ of 2-dimensional quadrics with center in terms of common conjugate systems (the condition that the conjugate system on a 2-dimensional quadric is a conjugate system on one of its deformations in $\mathbb{C}^3$ was known to Calapso for a decade, but the B transformation for quadrics eluded Calapso since the common conjugate system was best suited for this transformation only at the analytic level: to find the applicability correspondence one must renounce the correspondence of the common conjugate systems provided by the *Weingarten* (W) congruence in favor of the ACPIA).

In 1919-20 Cartan has shown in [9] (using mostly projective arguments and his exterior differential systems in involution and exteriorly orthogonal forms tools) that space forms of dimension n admit rich families of deformations (depending on $n(n-1)$ functions of one variable) in surrounding space forms of dimension $2n-1$, that such deformations admit lines of curvature (given by a canonical form of exteriorly orthogonal forms; thus they have flat normal bundle; since the lines of curvature on n-dimensional space forms (when they are considered by definition as quadrics in surrounding $(n+1)$-dimensional space forms) are undetermined, the lines of curvature on the deformation and their corresponding curves on the quadric provide the common conjugate system) and that the codimension $n-1$ cannot be lowered without obtaining rigidity as the deformation being the defining quadric.

In 1979, upon a suggestion from S. S. Chern and using Chebyshev coordinates on $\mathbb{H}^n(\mathbb{R})$ (the Cartan-Moore Theorem; they are the lines of curvature on and thus in bijective correspondence with deformations of $\mathbb{H}^n(\mathbb{R})$ in $\mathbb{R}^{2n-1}$) Tenenblat-Terng have developed in [16] the B transformation of $\mathbb{H}^n(\mathbb{R})$ in $\mathbb{R}^{2n-1}$ (and Terng in [17] has developed the *Bianchi Permutability Theorem* (BPT) for this B transformation).

In 1983 Berger, Bryant and Griffiths [2] proved (including by use of tools from algebraic geometry) in particular that Cartan's essentially projective arguments (including the exterior part of his exteriorly orthogonal forms tool) can be used to generalize his results to n-dimensional general quadrics with positive definite linear element (thus they can appear as quadrics in $\mathbb{R}^{n+1}$ or as space-like quadrics in $\mathbb{R}^n \times (i\mathbb{R})$) admitting rich families of deformations (depending on $n(n-1)$ functions of one variable) in surrounding Euclidean space $\mathbb{R}^{2n-1}$, that the codimension $n-1$ cannot be lowered without obtaining rigidity as the deformation being the defining quadric and that quadrics are the only Riemannian n-dimensional manifolds that admit a family of deformations in $\mathbb{R}^{2n-1}$ as rich as possible for which the exteriorly orthogonal forms tool (naturally appearing from the Gauß equations) can be applied.

The starting point of our investigation is the higher dimensional (infinitesimal) version of Bianchi's main four theorems on the theory of deformations of quadrics and Bianchi's treatment of the B transformation for paraboloids via conjugate systems.

Similarly to Bianchi's original (pre-Ivory affinity) approach to the deformation problem for quadrics, when he made a link between deformations of diagonal paraboloids and the sine-Gordon equation, we shall consider first paraboloids, since the higher dimensional version of the sine-Gordon equation (namely Terng's *generalized sine-Gordon equation* (GSGE)) together with its B transformation is already completed by Tenenblat-Terng and Terng has already completed the BPT in this case.

Again just like Bianchi considered *isothermic-conjugate* coordinates on the quadric (that is the second fundamental form is a multiple of the identity) coupled with the conjugate system common to the quadric and its deformation (the change between these conjugate systems provides the angle that satisfies the sine-Gordon equation and the ACPIA is best seen at the level of the initial isothermic-conjugate system on the quadric), we shall consider isothermic-coordinates on the quadric coupled with the conjugate system common to both the quadric and its deformation, since this interplay between the two systems of coordinates will make the deformation problem amenable to an attack strategy (the common conjugate system property provides the canonical form of exteriorly orthogonal forms). The B transformation will turn out to be similar in some aspects to that of Tenenblat-Terng (in fact at the level of the analytic computations it will obey the same equations) for (isotropic) quadrics without center, but of a more general nature and more importantly it will appear as a natural consequence of the ACPIA.

Once some knowledge about the *rigid motion provided by the Ivory affinity* (RMPIA) is drawn, we shall use this (just as Bianchi) to provide a simpler proof without using common conjugate systems.

All computations are local and assumed to be valid on their open domain of validity without further details; all functions have the assumed order of differentiability and are assumed to be invertible, non-zero, etc when required (for all practical purposes we can assume all functions to be analytic).

2. Confocal quadrics in canonical form

Consider the complexified Euclidean space

$$(\mathbb{C}^{n+1}, <.,.>),\ <x,y>:=x^Ty,\ |x|^2:=x^Tx,\ x,y\in\mathbb{C}^{n+1}$$

with standard basis $\{e_j\}_{j=1,\dots,n+1},\ e_j^Te_k=\delta_{jk}$.

Isotropic (null) vectors are those vectors v of length 0 ($|v|^2=0$); since most vectors are not isotropic we shall call a vector simply vector and we shall only emphasize isotropic when the vector is assumed to be isotropic. The same denomination will apply in other settings: for example we call quadric a non-degenerate quadric (a quadric projectively equivalent to the complex unit sphere).

A quadric $x\subset\mathbb{C}^{n+1}$ is given by the equation $Q(x):=\begin{bmatrix}x\\1\end{bmatrix}^T\begin{bmatrix}A & B\\B^T & C\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}=x^T(Ax+2B)+C=0,\ A=A^T\in\mathbf{M}_{n+1}(\mathbb{C}),\ B\in\mathbb{C}^{n+1},\ C\in\mathbb{C},\ \begin{vmatrix}A & B\\B^T & C\end{vmatrix}\neq 0.$

A metric classification of all (totally real) quadrics in $\mathbb{C}^{n+1}$ requires the notion of *symmetric Jordan* (SJ) canonical form of a symmetric complex matrix. The symmetric Jordan blocks are: $J_1:=0=0_{1,1}\in\mathbf{M}_1(\mathbb{C}),\ J_2:=f_1f_1^T\in\mathbf{M}_2(\mathbb{C}),\ J_3:=f_1e_3^T+e_3f_1^T\in\mathbf{M}_3(\mathbb{C}),\ J_4:=f_1\bar{f}_2^T+f_2f_2^T+\bar{f}_2f_1^T\in\mathbf{M}_4(\mathbb{C}),\ J_5:=f_1\bar{f}_2^T+f_2e_5^T+e_5f_2^T+\bar{f}_2f_1^T\in\mathbf{M}_5(\mathbb{C}),\ J_6:=f_1\bar{f}_2^T+f_2\bar{f}_3^T+f_3f_3^T+\bar{f}_3f_2^T+\bar{f}_2f_1^T\in\mathbf{M}_6(\mathbb{C})$, etc, where $f_j:=\frac{e_{2j-1}+ie_{2j}}{\sqrt{2}}$ are the standard isotropic vectors (at least the blocks $J_2,\ J_3$ were known to the classical geometers). Any symmetric complex matrix can be brought via conjugation with a complex rotation to the symmetric Jordan canonical form, that is a matrix block decomposition with blocks of the form $a_jI_p+J_p$; totally real quadrics are obtained for eigenvalues a_j of the quadratic part A defining the quadric being real or coming in complex conjugate pairs $a_j,\ \bar{a}_j$ with subjacent symmetric Jordan blocks of same dimension p. Just as the usual Jordan block $\sum_{j=1}^p e_je_{j+1}^T$ is nilpotent with e_{p+1} cyclic vector of order p, J_p is nilpotent with $\bar{f}_1$ cyclic vector of order p, so we can take square roots of SJ matrices without isotropic kernels ($\sqrt{aI_p+J_p}:=\sqrt{a}\sum_{j=0}^{p-1}\binom{\frac{1}{2}}{j}a^{-j}J_p^j,\ a\in\mathbb{C}^*,\ \sqrt{a}:=\sqrt{r}e^{i\theta}$ for $a=re^{2i\theta},\ 0<r,\ -\pi\le 2\theta<\pi$), two matrices with same SJ decomposition type (that is J_p is replaced with a polynomial in J_p) commute, etc.

The confocal family $\{x_z\}_{z\in\mathbb{C}}$ of a quadric $x_0\subset\mathbb{C}^{n+1}$ in canonical form (depending on as few constants as possible) is given in the projective space $\mathbb{CP}^{n+1}$ by the equation $Q_z(x_z):=\begin{bmatrix}x_z\\1\end{bmatrix}^T(\begin{bmatrix}A & B\\B^T & C\end{bmatrix}^{-1}-z\begin{bmatrix}I_{n+1} & 0\\0^T & 0\end{bmatrix})^{-1}\begin{bmatrix}x_z\\1\end{bmatrix}=0$, where

i) $A=A^T\in\mathbf{GL}_{n+1}(\mathbb{C})$ SJ, $B=0\in\mathbb{C}^{n+1},\ C=-1$ for *quadrics with center* (QC),

ii) $A=A^T\in\mathbf{M}_{n+1}(\mathbb{C})$ SJ, $\ker(A)=\mathbb{C}e_{n+1},\ B=-e_{n+1},\ C=0$ for *quadrics without center* (QWC) and

iii) $A=A^T\in\mathbf{M}_{n+1}(\mathbb{C})$ SJ, $\ker(A)=\mathbb{C}f_1,\ B=-\bar{f}_1,\ C=0$ for *isotropic quadrics without center* (IQWC).

From the definition one can see that the family of quadrics confocal to x_0 is the adjugate of the pencil generated by the adjugate of x_0 and Cayley's absolute $C(\infty)\subset\mathbb{CP}^n$ in the hyperplane at infinity; since Cayley's absolute encodes the Euclidean structure of $\mathbb{C}^{n+1}$ (it is the invariant set under rigid motions and homotheties of $\mathbb{C}^{n+1}:=\mathbb{CP}^{n+1}\setminus\mathbb{CP}^n$) the mixed metric-projective character of the confocal family becomes clear.

For Q(W)C spec(A) is unambiguous (does not change under rigid motions) but for IQWC it may change with $(p+1)$-roots of unity for the block of f_1 in A being J_p even under rigid motions which preserve the canonical form, so it is unambiguous up to $(p+1)$-roots of unity.

We have the diagonal Q(W)C respectively for $A=\Sigma_{j=1}^{n+1}a_j^{-1}e_je_j^T,\ A=\Sigma_{j=1}^{n}a_j^{-1}e_je_j^T$; the diagonal IQWC come in different flavors, according to the block of $f_1:\ A=J_p+\Sigma_{j=p+1}^{n+1}a_j^{-1}e_je_j^T$; in particular if $A=J_{n+1}$, then spec$(A)=\{0\}$ is unambiguous. Thus general quadrics are those for which all eigenvalues have geometric multiplicity 1; equivalently each eigenvalue has an only corresponding SJ block; in this case the quadric also admits elliptic coordinates.

There are continuous groups of symmetries which preserve the SJ canonical form for more than one SJ block corresponding to an eigenvalue, so from a metric point of view a metric classification according to the elliptic coordinates and continuous symmetries may be a better one.

With $R_z:=I_{n+1}-zA,\ z\in\mathbb{C}\setminus\mathrm{spec}(A)^{-1}$ the family of quadrics $\{x_z\}_z$ confocal to x_0 is given by $Q_z(x_z)=x_z^TAR_z^{-1}x_z+2(R_z^{-1}B)^Tx_z+C+zB^TR_z^{-1}B=0$. For $z\in\mathrm{spec}(A)^{-1}$ we obtain singular confocal quadrics; those with z^{-1} having geometric multiplicity 1 admit a singular set which is an $(n-1)$-dimensional quadric projectively equivalent to $C(\infty)$, so they will play an important rôle in the discussion of homographies $H\in\mathbf{PGL}_{n+1}(\mathbb{C})$ taking a confocal family into another one, since $H^{-1}(C(\infty)),\ C(\infty)$ respectively $C(\infty),\ H(C(\infty))$ will suffice to determine each confocal family.

The Ivory affinity is an affine correspondence between confocal quadrics and having good metric properties (it may be the reason why Bianchi calls it *affinity* in more than one language): it is given by $x_z = \sqrt{R_z}x_0 + C(z),\ C(z) := -(\frac{1}{2}\int_0^z(\sqrt{R_w})^{-1}dw)B$. Note that $C(z) = 0$ for QC, $= \frac{z}{2}e_{n+1}$ for QWC; for IQWC it is the Taylor series of $\frac{1}{2}\int_0^z(\sqrt{1-w})^{-1}dw$ at $z = 0$ with each monomial z^{k+1} replaced by $z^{k+1}J_p^k\bar{f}_1$, where J_p is the block of f_1 in A and thus a polynomial of degree p in z. Note $AC(z) + (I_{n+1} - \sqrt{R_z})B = 0 = (I_{n+1} + \sqrt{R_z})C(z) + zB$. Applying d to $Q_z(x_z) = 0$ we get $dx_z^TR_z^{-1}(Ax_z + B) = 0$, so the unit normal N_z is proportional to $\hat{N}_z := -2\partial_zx_z$. If $\mathbb{C}^{n+1} \ni x \in x_{z_1}, x_{z_2}$, then $\hat{N}_{z_j} = R_{z_j}^{-1}(Ax + B)$; using $R_z^{-1} - I_{n+1} = zAR_z^{-1},\ z_1R_{z_1}^{-1} - z_2R_{z_2}^{-1} = (z_1 - z_2)R_{z_1}^{-1}R_{z_2}^{-1}$ we get $0 = Q_{z_1}(x) - Q_{z_2}(x) = (z_1 - z_2)\hat{N}_{z_1}^T\hat{N}_{z_2}$, so two confocal quadrics cut each other orthogonally (Lamé). For general quadrics the polynomial equation $Q_z(x) = 0$ has degree $n + 1$ in z and it has multiple roots iff $0 = \partial_zQ_z(x) = |\hat{N}_z|^2$; thus outside the locus of isotropic normals elliptic coordinates (given by the roots $z_1, ..., z_{n+1}$ of the said equation) give a parametrization of $\mathbb{C}^{n+1}$ suited to confocal quadrics.

We have now some classical metric properties of the Ivory affinity: with $x_0^0, x_0^1 \in x_0,\ V_0^1 := x_z^1 - x_0^0$, etc the Ivory Theorem (preservation of length of segments between confocal quadrics) becomes $|V_0^1|^2 = |x_0^0 + x_0^1 - C(z)|^2 - 2(x_0^0)^T(I_{n+1} + \sqrt{R_z})x_0^1 + zC = |V_1^0|^2$; the preservation of lengths of rulings: $w_0^TAw_0 = w_0^T\hat{N}_0 = 0,\ w_z = \sqrt{R_z}w_0 \Rightarrow w_z^Tw_z = |w_0|^2 - zw_0^TAw_0 = |w_0|^2$; the symmetry of the TC: $(V_0^1)^T\hat{N}_0^0 = (x_0^0)^TA\sqrt{R_z}x_0^1 - B^T(x_z^0 + x_z^1 - C(z)) + C = (V_1^0)^T\hat{N}_0^1$; the preservation of angles between segments and rulings: $(V_0^1)^Tw_0^0 + (V_1^0)^Tw_z^0 = -z(\hat{N}_0^0)^Tw_0^0 = 0$; the preservation of angles between rulings: $(w_0^0)^Tw_z^1 = (w_0^0)^T\sqrt{R_z}w_0^1 = (w_z^0)^Tw_0^1$; the preservation of angles between polar rulings: $(w_0^0)^TA\hat{w}_0^0 = 0 \Rightarrow (w_z^0)^T\hat{w}_z^0 = (w_0^0)^T\hat{w}_0^0 - z(w_0^0)^TA\hat{w}_0^0 = (w_0^0)^T\hat{w}_0^0$.

All complex quadrics are affine equivalent to either the unit sphere $X \subset \mathbb{C}^{n+1},\ |X|^2 = 1$ or to the equilateral paraboloid $Z \subset \mathbb{C}^{n+1},\ Z^T(I_{1,n}Z - 2e_{n+1}) = 0$, so a parametrization with regard to these two quadrics is in order: $x_0 = (\sqrt{A})^{-1}X$ for QC, $x_0 = (\sqrt{A + e_{n+1}e_{n+1}^T})^{-1}Z$ for QWC (for this reason from a canonical metric point of view (that is we are interested in a simplest form of $|dx_0|^2$) we should rather require that A^{-1} or $(A + e_{n+1}e_{n+1}^T)^{-1}$ is SJ).

For IQWC such a parametrization fails because of the isotropic $\ker(A)$; however the computations between confocal quadrics involving the Ivory affinity reveal a natural parametrization of IQWC which is again an affine transformation of Z.

Consider a canonical IQWC $x_0^T(Ax_0 - 2\bar{f}_1) = 0,\ \ker(A) = \mathbb{C}f_1,\ A = J_p \oplus ...$ SJ. We are looking for a linear map $L \in \mathbf{GL}_{n+1}(\mathbb{C})$ such that $x_0 = LZ$, equivalently $L^TAL = e^{2a}I_{1,n},\ I_{1,n} := I_{n+1} - e_{n+1}e_{n+1}^T,\ L^T\bar{f}_1 = e^{2a}e_{n+1}$. Replacing L with $L(e^{-a}I_{1,n} + e^{-2a}e_{n+1}e_{n+1}^T)$ we can make $a = 0$. Thus $Le_{n+1} = f_1,\ L^T(A + \bar{f}_1\bar{f}_1^T)L = I_{n+1}$, so $L^{-1} = R^T\sqrt{A + \bar{f}_1\bar{f}_1^T},\ R^TR = I_{n+1}$ with $Re_{n+1} = \sqrt{A + \bar{f}_1\bar{f}_1^T}f_1$ (note that Re_{n+1} has, as required, length 1). Once $R \in \mathbf{O}_{n+1}(\mathbb{C})$ with the above property is found, L thus defined satisfies $L^T\bar{f}_1 = e_{n+1}$ and thus $L^TAL = I_{1,n}$. L with the above properties is unique up to rotations fixing e_{n+1} in its domain and a canonical choice of R will reveal itself from a SJ canonical form when doing computations on confocal quadrics. We have $LL^T = (A + \bar{f}_1\bar{f}_1^T)^{-1},\ I_{1,n}L^{-1}\sqrt{R_z}L = I_{1,n}L^{-1}\sqrt{R_z}LI_{1,n} = L^{-1}\sqrt{R_z}L - e_{n+1}\bar{f}_1\sqrt{R_z}L = L^TA\sqrt{R_z}L = I_{1,n}\sqrt{I_{n+1} - zL^TA^2L} =: I_{1,n}\sqrt{R'_z},\ A' := L^TA^2L,\ \ker(A') = \mathbb{C}e_{n+1} \oplus \mathbb{C}L^{-1}(A + \bar{f}_1\bar{f}_1^T)^{-1}f_1 = \mathbb{C}e_{n+1} \oplus \mathbb{C}L^Tf_1$; choose R which makes A' SJ. Note that we can take for QWC $L := (\sqrt{A + e_{n+1}e_{n+1}^T})^{-1},\ A' := A,\ \ker(A') = \mathbb{C}e_{n+1}$, so IQWC can be regarded as metrically degenerated QWC. Note that $e_{n+1}^TL^{-1}\sqrt{R_z}L = (-I_{1,n}L^{-1}C(z) + e_{n+1})^T$; this can be confirmed analytically by differentiating with respect to z and using $(L^T)^{-1} = AL - Be_{n+1}^T$ and will imply the symmetry of the TC, but since we have already proved the symmetry of the TC, we can use this to imply the previous. Thus $L^{-1}x_z = L^{-1}\sqrt{R_z}LZ + L^{-1}C(z) = I_{1,n}\sqrt{R'_z}Z + e_{n+1}(-I_{1,n}L^{-1}C(z) + e_{n+1})^TZ + L^{-1}C(z),\ (x_z^1 - x_0^0)^T\hat{N}_0^0 = (L^{-1}x_z^1 - Z_0)^T(I_{1,n}Z_0 - e_{n+1}) = Z_0^TI_{1,n}\sqrt{R'_z}Z_1 + (Z_0 + Z_1)^T(I_{1,n}L^{-1}C(z) - e_{n+1}) - e_{n+1}^TL^{-1}C(z)$. Note that for IQWC $|I_{1,n}L^{-1}C(z)|^2 = 2e_{n+1}^TL^{-1}C(z)$, so $L^{-1}C(z)$ lies itself on Z (also in this case since $\bar{f}_1^TJ_p^k\bar{f}_1 = \delta_{k\ p-1}$ we have $e_{n+1}^TL^{-1}C(z) =$

$\bar{f}_1^T C(z) = \binom{-\frac{1}{2}}{p-1}\frac{(-z)^p}{-2p}$, so e_{n+1}^T picks up the highest power of z in $L^{-1}C(z)$). To see this we need $0 = |L^{-1}C(z) - \bar{f}_1^T C(z) e_{n+1}|^2 - 2\bar{f}_1^T C(z) = C(z)^T (LL^T)^{-1} C(z) - (\bar{f}_1^T C(z))^2 - 2\bar{f}_1^T C(z) = C(z)^T A C(z) - 2\bar{f}_1^T C(z)$; using $AC(z) = (I_{n+1} - \sqrt{R_z})\bar{f}_1$ and $(I_{n+1} + \sqrt{R_z})C(z) = z\bar{f}_1$ it is satisfied.

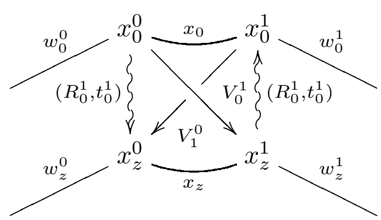
Note also that as needed later we have $(L^T L)^{-1} = A' - I_{1,n}L^{-1}Be_{n+1}^T - e_{n+1}(I_{1,n}L^{-1}B)^T + |B|^2 e_{n+1}e_{n+1}^T,\ |\hat{N}_0|^2 = |(L^T)^{-1}(I_{1,n}Z - e_{n+1})|^2 = Z^T A' Z + 2Z^T I_{1,n}L^{-1}B + |B|^2$,
$(I_{n+1} + \sqrt{R_z'})I_{1,n}L^{-1}C(z) = I_{1,n}L^{-1}(I_{n+1} + \sqrt{R_z})C(z) = -zI_{1,n}L^{-1}B$.

3. The Bianchi Truths

We hold these (a-priori not-so-evident) Bianchi Truths to be self-evident:

I (existence and inversion of the Bäcklund transformation for quadrics and the applicability correspondence provided by the Ivory affinity)

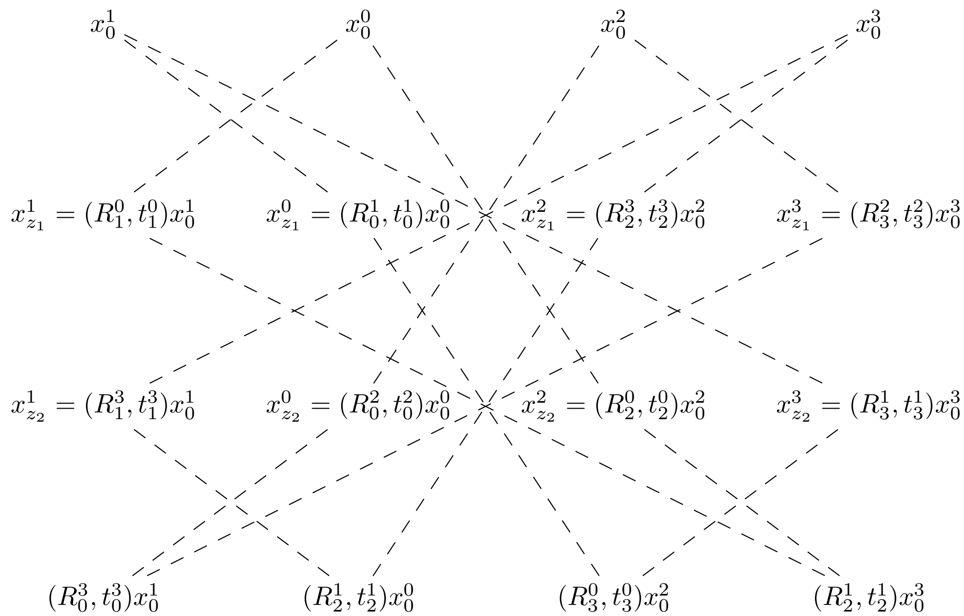
That any deformation $x^0 \subset \mathbb{C}^{2n-1}$ of an n-dimensional sub-manifold $x_0^0 \subseteq x_0$ ($x_0 \subset \mathbb{C}^{n+1} \subset \mathbb{C}^{2n-1}$ being a quadric) appears as a focal sub-manifold of an $(\frac{n(n-1)}{2}+1)$-dimensional family of Weingarten congruences, whose other focal sub-manifolds $x^1 = B_z(x^0)$ are applicable, via the Ivory affinity between confocal quadrics, to sub-manifolds x_0^1 in the same quadric x_0. The determination of these sub-manifolds requires the integration of a family of Riccati equations depending on the parameter z (we ignore for simplicity the dependence on the initial value data in the notation B_z). Moreover, if we compose the inverse of the rigid motion provided by the Ivory affinity (RMPIA) (R_0^1, t_0^1) with the rolling of x_0^0 on x^0, then we obtain the rolling of x_0^1 on x^1 and x^0 reveals itself as a B_z transform of x^1;



$$(R_j, t_j)(x_0^j, dx_0^j) := (R_j x_0^j + t_j, R_j dx_0^j) = (x^j, dx^j),\ (R_j, t_j) \in \mathbf{O}_{2n-1}(\mathbb{C}) \ltimes \mathbb{C}^{2n-1},\ j = 0, 1,$$
$$(R_0^1, t_0^1) = (R_1, t_1)^{-1}(R_0, t_0). \tag{1}$$

II (Bianchi Permutability Theorem)

That if $x^1 = B_{z_1}(x^0),\ x^2 = B_{z_2}(x^0)$, then one can find only by algebraic computations a sub-manifold $B_{z_2}(x^1) = x^3 = B_{z_1}(x^2)$; thus $B_{z_2} \circ B_{z_1} = B_{z_1} \circ B_{z_2}$ and once all B transforms of the seed x^0 are found, the B transformation can be iterated using only algebraic computations;



$$
\begin{aligned}
&(R_j, t_j)(x^j, dx^j) = (x_0^j, dx_0^j),\ j = 0, ..., 3,\\
(2)\qquad &(R_j^k, t_j^k) = (R_k, t_k)^{-1}(R_j, t_j),\ (j,k) = (0,1), (0,2), (1,3), (2,3),\\
&(R_0^1, t_0^1)(R_2^0, t_2^0) = (R_3^0, t_3^0) = (R_3^1, t_3^1)(R_2^3, t_2^3),\ (R_1^0, t_1^0)(R_3^1, t_3^1) = (R_2^1, t_2^1) = (R_2^0, t_2^0)(R_3^2, t_3^2).
\end{aligned}
$$

$$
\begin{tikzcd}
x^2 \arrow[r, leftrightarrow, "B_{z_1}"] \arrow[d, leftrightarrow, "B_{z_2}"'] \arrow[dr, phantom, "\#"] & x^3 \arrow[d, leftrightarrow, "B_{z_2}"] \\
x^0 \arrow[r, leftrightarrow, "B_{z_1}"'] & x^1
\end{tikzcd}
$$

III (existence of 3-Möbius moving configurations $\mathcal{M}_3$)

That if $x^1 = B_{z_1}(x^0)$, $x^2 = B_{z_2}(x^0)$, $x^4 = B_{z_3}(x^0)$ and by use of the Bianchi Permutability Theorem one finds $B_{z_3}(x^2) = x^6 = B_{z_2}(x^4)$, $B_{z_1}(x^4) = x^5 = B_{z_3}(x^1)$, $B_{z_2}(x^1) = x^3 = B_{z_1}(x^2)$, $B_{z_3}(x^3) = x'^7 = B_{z_2}(x^5)$, $B_{z_1}(x^6) = x''^7 = B_{z_3}(x^3)$, $B_{z_2}(x^5) = x'''^7 = B_{z_1}(x^6)$, then $x'^7 = x''^7 = x'''^7 =: x^7$; thus once all B transforms of the seed x^0 are found, the B transformation can be further iterated using only algebraic computations;

$$
\begin{tikzcd}
& x^6 \arrow[rr, leftrightarrow, "B_{z_1}"] \arrow[dl, leftrightarrow, "B_{z_2}"'] \arrow[dd, leftrightarrow, "B_{z_3}"] & & x^7 \arrow[dl, leftrightarrow, "B_{z_2}"] \arrow[dd, leftrightarrow, "B_{z_3}"] \\
x^4 \arrow[rr, leftrightarrow, "B_{z_1}"'] \arrow[dd, leftrightarrow, "B_{z_3}"'] & & x^5 \arrow[dd, leftrightarrow, "B_{z_3}"] & \\
& x^2 \arrow[rr, leftrightarrow, "B_{z_1}"'] & & x^3 \arrow[dl, leftrightarrow, "B_{z_2}"] \\
x^0 \arrow[rr, leftrightarrow, "B_{z_1}"'] \arrow[ur, leftrightarrow, "B_{z_2}"] & & x^1 &
\end{tikzcd}
$$

IV (Hazzidakis transformation)

That if an n-dimensional sub-manifold $x^0 \subset \mathbb{C}^{2n-1}$ is applicable to a sub-manifold $x^0_0 \subseteq x_0$ and the homography $H \in \mathbf{PGL}_{n+1}(\mathbb{C})$ takes the confocal family x_z to another confocal family $\tilde{x}_{\tilde{z}},\ \tilde{z} = \tilde{z}(z),\ \tilde{z}(0) = 0$, then one infinitesimally knows a sub-manifold $\tilde{x}^0 = H(x^0)$ (that is one knows the first and second fundamental forms), called the Hazzidakis (H) transform of x^0 and applicable to a sub-manifold $\tilde{x}^0_0 \subseteq \tilde{x}_0$. Moreover the H transformation commutes with the B transformation ($H \circ B_z = B_{\tilde{z}} \circ H$) and the $B_{\tilde{z}}(\tilde{x}^0)$ transforms can be algebraically recovered from the knowledge of $\tilde{x}^0$ and $B_z(x^0)$.

Keeping an eye on

0 (The Method of Archimedes)

'... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge'

we ask ourselves what happens when the *'mechanical method'* in question is the rolling. In this case *'first'* means that the seed x^0 is in the particular position of actually coinciding with $x^0_0 \subseteq x_0 \subset \mathbb{C}^{n+1} \subset \mathbb{C}^{2n-1}$: for $n = 2$ the leaves become rulings on a confocal quadric (Bianchi and Lie; in fact it is this small observation of Lie's for the pseudo-sphere that proved to be the essential tool used by Bianchi to develop the theory of deformations of quadrics, but the classical geometers did not investigate what becomes of Lie's powerful *facets* (pairs of points and planes passing through those points) approach when the leaf degenerates from a surface to a ruling).

Thus the natural conjecture appears that the B transforms x^1 (leaves) of x^0 (seed) are rulings on quadrics x_z confocal to x_0 when $x^0 = x^0_0 \subset x_0$. An a-priori argument in favor of rulings on quadrics x_z confocal to x_0 being the leaves when $x^0 = x^0_0$ is the fact that in this case the leaves should be sub-manifolds of x_z; thus their tangent spaces should be contained both in the tangent bundle of x_z and in the distribution of facets; since facets intersect tangent spaces of x_z only along rulings of x_z the leaves should be rulings. Also rulings preserve their length under the Ivory affinity, so they should be naturally chosen by the RMPIA (the discretization of the infinitesimal version of the ACPIA).

Due to the symmetries (in the normal bundle) of the rolling we get the $\mathbf{O}_{n-1}(\mathbb{C}) \times \mathbf{O}_{n-1}(\mathbb{C})$ symmetries of the facets in the tangency configuration and due to the initial value data of the differential system subjacent to the B transformation we get the fact that for the deformation problem of n-dimensional quadrics in $\mathbb{C}^{2n-1}$ no functional information is allowed in the normal bundle.

Since by rolling the position of the leaf x^1 relative to the seed x^0 is the same as the position of x^1_z relative to x^0_0, we get the fact that the tangential part of dx^1 (relative to x^0) is the same as the tangential part of dx^1_z (relative to x^0_0). Thus with $N^0 := [N^0_{n+1}\ \dots\ N^0_{2n-1}]$ orthonormal normal frame of x^0 and N^0_0 unit normal of x^0_0 we have $|dx^1|^2 = |(dx^1)^\top|^2 + |(dN^0)^T(x^1 - x^0)|^2,\ |dx^1_z|^2 = |(dx^1_z)^\top|^2 + |(dN^0_0)^T(x^1_z - x^0_0)|^2$ and the ACPIA $|dx^1|^2 = |dx^1_0|^2$ becomes

$$|dx^1_0|^2 - |dx^1_z|^2 = |\begin{bmatrix} -i(dN^0_0)^T(x^1_z - x^0_0) \\ -(dN^0)^T(x^1 - x^0) \end{bmatrix}|^2 \tag{3}$$

(we augment the $(n-1)$-column 1-form $-(dN^0)^T(x^1 - x^0)$ with the entry $-i(dN^0_0)^T(x^1_z - x^0_0)$ on the first position to get an n-column 1-form; thus the second fundamental forms are naturally joined).

Equation (3) will turn out to be essential, since by the metric properties of the Ivory affinity the *left hand side* (lhs) will be an n-dimensional symmetric quadratic form $|\omega_1|^2$ which is a square for proper choice of coordinates, just like the *right hand side* (rhs), so one obtains an equality $\omega_1 = R\omega_0$ of n-column 1-forms (involving an a-priori arbitrary rotation matrix $R \subset \mathbf{O}_n(\mathbb{C})$).

However, at least for the first draft a coordinate-independent approach seems to stop here: to continue two sets of coordinates must be used in order to take full advantage of (3): a set of coordinates to express in a simplest form the lhs and other for the rhs of (3), while keeping in mind meaningful formulae for changes of coordinates (the change of Christoffel symbols will turn

out to be the main problem). For the lhs we need to take into consideration only two cases of quadrics: the equilateral paraboloid (when the parametrization is the one that realizes x_0 as a graph) and the unit sphere (when the parametrization is given by the stereo-graphical projection; it is the projective transformation of the graph coordinates on the equilateral paraboloid under the projective transformation which takes the point at infinity on the e_{n+1}-axis to the north pole); these parametrizations have the advantage of providing an isothermic-conjugate system on x_0. For the rhs we shall use conjugate systems common to both x_0^0 and x^0, since they will considerably simplify the computations and will naturally fit for the deformation problem for higher dimensional quadrics, just as for 2-dimensional quadrics.

The investigation of the first three Bianchi Truths respectively boil down when $x^0 = x_0^0$ to the investigation of the *first, second and third iterations of the tangency configuration* (TC, SITC, TITC). The Bianchi Truths I - IV become respectively the Bianchi I - IV Theorems on confocal quadrics (some of the metric properties of the Ivory affinity between confocal quadrics and beyond the Ivory Theorem on the preservation under the Ivory affinity of lengths of segments between confocal quadrics were already known to other authors; for example Henrici's construction of the articulated hyperbolic paraboloid uses preservation of lengths of rulings under the Ivory affinity).

Note that for $n = 2$ the Bianchi Truth III uses a theorem of Menelaus (in itself a co-cycle theorem), which is equivalent to the infinitesimal associativity of a loop group action.

Since the dimensionality of the space of leaves should be independent of the shape of the seed (and thus should equal the dimensionality of the space of rulings), the two coincide ($2n-3 = \frac{n(n-1)}{2}$) without requiring multiplicities of rulings only for $n = 2, 3$.

Note that the SITC for an edge of the Bianchi quadrilateral being infinitesimal infinitesimally describes the B transformation; thus the SITC encodes all necessary and sufficient information needed to prove the Bianchi Truth I. A similar statement holds for the BPT and the TITC, so the first three iterations of the tangency configuration contain all necessary algebraic information needed to develop the theory of deformations of quadrics, as expected (by discretization each iteration corresponds to a derivative, so each iteration encodes respectively the information in the tangent space, the *Gauß-Weingarten* (GW) equations and the *Gauß-Codazzi-Mainardi(-Peterson)-Ricci* (G-CMP-R) equations).

While a simple conjugation trick in the BPT provides a tool to obtain totally real deformations of the same metric type and in the same Lorentz surrounding space $\mathbb{R}^m \times (i\mathbb{R})^{2n-1-m},\ 0 \le m \le 2n-1$ as the seed from totally real seed without worrying about the intermediary leaves, a full discussion of the cases when both the leaf and the seed are totally real (thus they must be situated in the same Lorentz space) is not completed even for $n = 2$.

4. The deformation problem for quadrics via common conjugate systems

First we shall recall the notion of deformations of quadrics with common conjugate system and non-degenerate joined second fundamental forms, appearing in one of our previous notes concerning Peterson's deformations of higher dimensional quadrics.

Consider the complexified Euclidean space

$$(\mathbb{C}^m, <.,.>),\ <x, y>:= x^T y,\ |x|^2 := x^T x,\ x, y \in \mathbb{C}^m$$

with standard basis $\{e_j\}_{j=1,...,m},\ e_j^T e_k = \delta_{jk}$ (we shall use $m = n, n+1, 2n-1$).

We shall always have Latin indices $j, k, l, ... \in \{1, ..., n\}$ (including for differentiating respectively with respect to $u^j,\ u^k,\ u^l, ...$), Greek ones $\alpha, \beta, \gamma, ... \in \{n+1, ..., 2n-1\}$ for $x \subset \mathbb{C}^{2n-1}$ (or $\alpha = 0$ for $x_0 \subset \mathbb{C}^{n+1}$) and mute summation for upper and lower indices when clear from the context; also we shall preserve the classical notation d^2 for the tensorial (symmetric) second derivative and we shall use $d\wedge$ for the exterior (antisymmetric) derivative; thus $d \wedge d = 0$.

For $n \ge 3$ consider the n-dimensional sub-manifold

$$x = x(u^1, u^2, ..., u^n) \subset \mathbb{C}^{n+p},\ du^1 \wedge du^2 \wedge ... \wedge du^n \neq 0,\ p = 1, n-1$$

such that the tangent space at any point of x is not isotropic (the scalar product induced on it by the Euclidean one on $\mathbb{C}^{n+p}$ is not degenerate; this assures the existence of orthonormal normal frames). We have the normal frame $N := [N_{n+1} \ \ N_{n+p}]$, $N^TN = I_p$, the first $|dx|^2 = g_{jk}du^j \odot du^k$ and second $d^2x^TN = [h^{n+1}_{jk}du^j \odot du^k \ \ ... \ \ h^{n+p}_{jk}du^j \odot du^k]$ fundamental forms, the Christoffel symbols $\Gamma^l_{jk} = \frac{g^{lm}}{2}[(g_{jm})_k + (g_{km})_j - (g_{jk})_m]$, the Riemann curvature $R_{jmkl} = g_{mp}R^p_{jkl} = g_{mp}[(\Gamma^p_{jk})_l - (\Gamma^p_{jl})_k + \Gamma^q_{jk}\Gamma^p_{ql} - \Gamma^q_{jl}\Gamma^p_{qk}]$ tensor, the normal connection $N^TdN = \{n^\alpha_{\beta j}du^j\}_{\alpha,\beta=n+1,...,n+p}$, $n^\alpha_{\beta j} = -n^\beta_{\alpha j}$ and the curvature $r^\beta_{\alpha jk} = (n^\beta_{\alpha j})_k - (n^\beta_{\alpha k})_j + n^\gamma_{\alpha j}n^\beta_{\gamma k} - n^\gamma_{\alpha k}n^\beta_{\gamma j}$ tensor of the normal bundle.

We have the *Gauß-Weingarten* (GW) equations

$$x_{jk} = \Gamma^l_{jk}x_l + h^\alpha_{jk}N_\alpha,\ (N_\alpha)_j = -h^\alpha_{jk}g^{kl}x_l + n^\beta_{j\alpha}N_\beta$$

and their integrability conditions $x_{jkl} = x_{jlk}$, $(N_\alpha)_{jk} = (N_\alpha)_{kj}$, from where one obtains by taking the tangential and normal components (using $-(g^{jk})_l = g^{jm}\Gamma^k_{ml} + g^{km}\Gamma^j_{ml}$ and the GW equations themselves) the *Gauß-Codazzi-Mainardi(-Peterson)-Ricci* (G-CMP-R) equations

$$R_{jmkl} = \sum_\alpha (h^\alpha_{jk}h^\alpha_{lm} - h^\alpha_{jl}h^\alpha_{km}),\ (h^\alpha_{jk})_l - (h^\alpha_{jl})_k + \Gamma^m_{jk}h^\alpha_{ml} - \Gamma^m_{jl}h^\alpha_{mk} + h^\beta_{jk}n^\alpha_{\beta l} - h^\beta_{jl}n^\alpha_{\beta k} = 0,$$

$$r^\beta_{\alpha jk} = h^\alpha_{jl}g^{lm}h^\beta_{mk} - h^\alpha_{kl}g^{lm}h^\beta_{mj}.$$

If we have conjugate system $h^\alpha_{jk} =: \delta_{jk}h^\alpha_j$, then the above equations become:

$$R_{jkjk} = -R_{jkkj} = \sum_\alpha h^\alpha_j h^\alpha_k,\ (h^\alpha_j)_k = \Gamma^j_{jk}h^\alpha_j - \Gamma^k_{jj}h^\alpha_k - h^\beta_j\eta^\alpha_{\beta k},\ j \neq k,\ R_{jklm} = 0 \text{ otherwise},$$

$$\Gamma^l_{jk}h^\alpha_l = \Gamma^k_{jl}h^\alpha_k,\ j,k,l \text{ distinct},\ r^\beta_{\alpha jk} = (h^\alpha_j h^\beta_k - h^\beta_j h^\alpha_k)g^{jk}. \tag{4}$$

In particular for lines of curvature parametrization ($g_{jk} = \delta_{jk}g_{jk}$) we have flat normal bundle, so one can choose up to multiplication on the right by a constant matrix $\in \mathbf{O}_p(\mathbb{C})$ normal frame N with zero normal connection $N^TdN = 0$.

Consider now $x \subset \mathbb{C}^{2n-1}$ deformation of $x_0 \subset \mathbb{C}^{n+1}$ (that is $|dx_0|^2 = |dx|^2$) with common conjugate system $(u^1, ..., u^n)$ and non-degenerate joined second fundamental forms (that is $[d^2x_0^TN_0 \ \ d^2x^TN]$ is a symmetric quadratic $\mathbb{C}^n$-valued form which contains only $(du^j)^2$ terms for N_0 normal field of x_0 and $N = [N_{n+1} \ \ ... \ \ N_{2n-1}]$ normal frame of x and the dimension n cannot be lowered for $(u^1, ..., u^n)$ in an open dense set).

By the argument of Cartan's reduction of exteriorly orthogonal forms to the canonical form such coordinates $\{u^j\}_j$ exist for real deformations $\subset \mathbb{R}^{2n-1}$ of imaginary quadrics $\subset \mathbb{R}^n \times i\mathbb{R}$ (for such cases also all computations in the deformation problem will be real; see [2]). However, since we shall derive completely integrable differential systems (systems in involution) for the deformation problem for quadrics, the dimensionality of the space of deformations of quadrics remains the same (namely solution depending on $n(n-1)$ functions of one variable) in the complex setting also (the Cartan characters remain the same).

The common conjugate system with non-degenerate joined second fundamental forms assumption amounts to the vectors $h_j := [ih^0_j \ \ h^{n+1}_j \ \ ... \ \ h^{2n-1}_j]^T$ being linearly independent. From the Gauß equations we obtain $h^0_jh^0_k = R_{jkjk} = \sum_\alpha h^\alpha_j h^\alpha_k,\ j \neq k \Leftrightarrow h_j^Th_k = \delta_{jk}|h_j|^2$; thus the vectors $h_j \subset \mathbb{C}^n$ are further orthogonal, which prevents them from being isotropic (should one of them be isotropic, by a rotation of $\mathbb{C}^n$ one can make it f_1 ($\mathbf{O}_n(\mathbb{C})$ acts transitively on the isotropic cone) and after subtracting suitable multiples of f_1 from the remaining ones by another rotation of $\mathbb{C}^n$ the remaining ones linear combinations of $e_3, ..., e_n$, so we would have $n-1$ linearly independent orthogonal vectors in $\mathbb{C}^{n-2}$, a contradiction), so $\mathbf{a}_j := |h_j| \neq 0,\ h_j =: \mathbf{a}_jv_j,\ [v_1 \ \ ... \ \ v_n] \subset \mathbf{O}_n(\mathbb{C})$.

Note that the linear element must satisfy the condition

$$\Gamma^l_{jk} = 0,\ j,k,l \text{ distinct}$$

and deformations $x \subset \mathbb{C}^{2n-1}$ of $x_0 \subset \mathbb{C}^{n+1}$ with common conjugate system and non-degenerate joined second fundamental forms are in bijective correspondence with solutions of the differential system

$$(\log \mathbf{a}_j)_k = \Gamma^j_{jk},\ j \neq k,\ \sum_j \frac{(h^0_j)^2}{\mathbf{a}_j^2} + 1 = 0. \tag{5}$$

Once a solution of this system is known, one finds the second fundamental form of x and then one finds x by the integration of a Ricatti equation and quadratures (the Gauß-Bonnet(-Peterson) Theorem).

Note that with $\gamma_{jk} := \Gamma^k_{jj}\frac{h^0_k}{h^0_j},\ j \neq k$ the condition

$$(\gamma_{jk})_j = (\gamma_{kj})_k = -2\gamma_{jk}\gamma_{kj},\ (\gamma_{lj})_k = 2(\gamma_{lj}\gamma_{lk} - \gamma_{lk}\gamma_{kj} - \gamma_{lj}\gamma_{jk}),\ j,\ k,\ l \text{ distinct} \tag{6}$$

is necessary and sufficient to get a maximal ($(n-1)$-dimensional) family of Peterson's deformations x of x_0 with common conjugate system $(u^1, ..., u^n)$ and non-degenerate joined second fundamental forms; it is invariant under transformations of the variables u^j into themselves, so it is a statement about curves of coordinates $(u^1, ..., \hat{u}^j, ..., u^n)$ =ct, $j = 1, ..., n$.

4.1. **Deformations of (isotropic) quadrics without center.**
For the specific computations of deformations of quadrics we shall use the convention $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ with 0 on the $(n+1)^{\text{th}}$ component; thus for example we can multiply $(n+1, n+1)$-matrices with n-column vectors and similarly one can extend (n,n) matrices to $(n+1, n+1)$ matrices with zeroes on the last column and row. The converse is also valid: an $(n+1, n+1)$ matrix with zeroes on the last column and row (or multiplied on the left with an n-row vector and on the right with an n-column vector) will be considered as an (n,n)-matrix.

With $V := \sum_{k=1}^n v^k e_k = [v^1 \ \ ... \ \ v^n]^T$ consider the complex equilateral paraboloid $Z = Z(v^1, ..., v^n) = V + \frac{|V|^2}{2}e_{n+1}$.

We have the (I)QWC $x_0 := LZ,\ L \in \mathbf{GL}_{n+1}(\mathbb{C})$ (recall $L := (\sqrt{A + e_{n+1}e_{n+1}^T})^{-1},\ \ker(A) = \mathbb{C}e_{n+1}, A$ SJ, $B = -e_{n+1}$ for QWC and $Le_{n+1} = f_1,\ L^T(A + \bar{f}_1\bar{f}_1^T)L = I_{n+1},\ A' := L^TA^2L$ SJ for $\ker(A) = \mathbb{C}f_1, A$ SJ, $B = -\bar{f}_1$ in the case of IQWC) with linear element, unit normal, second fundamental form and Christoffel symbols $|dx_0|^2 = dV^TL^TLdV + (V^TdV)^2|Le_{n+1}|^2 + 2(Le_{n+1})^TLdV(V^TdV),\ N_0 = \frac{(L^T)^{-1}V+B}{\sqrt{H}},\ N_0^Td^2x_0 = -\frac{|dV|^2}{\sqrt{H}},\ H := |(L^T)^{-1}V - B|^2 = V^TA'V + 2V^TL^{-1}B + |B|^2,\ \tilde{\Gamma}^l_{jk} = 0,\ j \neq k,\ \tilde{\Gamma}^k_{jj} = \frac{\partial \log\sqrt{H}}{\partial v^k}$. Note that we have a distinguished tangent vector field $\mathcal{V}_0 := \sum_{k=1}^n \frac{\partial \log \sqrt{H}}{\partial v^k}x_{0v^k} = (x_{0v^jv^j})^\top = -\frac{1}{\sqrt{H}}N_0 + Le_{n+1}$; it has the property $dx_{0v^j} = \mathcal{V}_0dv^j - N_0dN_0^Tx_{0v^j}$.

The condition (6) that $(v^1, ..., v^n)$ are common to a Peterson's $(n-1)$-dimensional family of deformations becomes $\frac{\partial^2 \log\sqrt{H}}{\partial v^j\partial v^k} = -2\frac{\partial \log\sqrt{H}}{\partial v^j}\frac{\partial\log\sqrt{H}}{\partial v^k},\ j \neq k \Leftrightarrow e_j^TA'e_k = 0,\ j \neq k$, so the first (n,n) entries of A' must be a diagonal matrix.

Because $(v^1, ..., v^n)$ are isothermal-conjugate and $(u^1, .., u^n)$ are conjugate on x_0, the Jacobian $\frac{\partial(v^1,...,v^n)}{\partial(u^1,...,u^n)}$ has orthogonal columns, so with $\lambda_j := |\frac{\partial V}{\partial u^j}|,\ \Lambda := [\lambda_1 \ \ ... \ \ \lambda_n]^T,\ \delta := \mathrm{diag}[du^1 \ \ ... \ \ du^n]$ we have $dV = R\delta\Lambda,\ R \subset \mathbf{O}_n(\mathbb{C})$. Multiplying the formula for the change of Christoffel symbols $\frac{\partial v^c}{\partial u^l}\Gamma^l_{jk} = \frac{\partial^2 v^c}{\partial u^j\partial u^k} + \frac{\partial v^a}{\partial u^j}\frac{\partial v^b}{\partial u^k}\tilde{\Gamma}^c_{ab} = \frac{\partial^2 v^c}{\partial u^j \partial u^k} + \lambda_j^2\delta_{jk}\frac{\partial \log\sqrt{H}}{\partial v^c}$ on the left with $\frac{\partial v^c}{\partial u^p}$ and summing after c we obtain $\Gamma^p_{jk} = \lambda_p^{-2}(\sum_c \frac{\partial v^c}{\partial u^p}\frac{\partial^2 v^c}{\partial u^j\partial u^k} + \lambda_j^2\delta_{jk}(\log\sqrt{H})_p) = \delta_{pk}(\log\lambda_k)_j + \delta_{jk}\frac{\lambda_j^2}{\lambda_p^2}(\log\frac{\sqrt{H}}{\lambda_j})_p + \delta_{pj}(\log\lambda_j)_k$, so $\Gamma^j_{jk} = (\log\lambda_j)_k,\ \Gamma^k_{jj} = \frac{\lambda_j^2}{\lambda_k^2}(\log\frac{\sqrt{H}}{\lambda_j})_k,\ j \neq k,\ \Gamma^j_{jj} = (\log(\lambda_j\sqrt{H}))_j$. We have $h^0_j = -\frac{\lambda_j^2}{\sqrt{H}}$; since $(\log\lambda_j)_k = \Gamma^j_{jk} = (\log\mathbf{a}_j)_k,\ j \neq k$ we get $\lambda_j = \phi_j(u^j)\mathbf{a}_j$; after a change of the u^j variable into itself we can make $\lambda_j = \mathbf{a}_j,\ j = 1, ..., n$, so from (5) $|\Lambda|^2 = -H$ and $\Lambda^Td\Lambda = -dV^T(A'V + L^{-1}B) = -\Lambda^T\delta R^T(A'V + L^{-1}B)$.

Imposing the compatibility condition $R^Td\wedge$ on $dV = R\delta\Lambda$ we get $R^TdR\wedge\delta\Lambda - \delta\wedge d\Lambda = 0$, or $(\lambda_j)_k = e_j^TR^TR_je_k\lambda_k,\ j\neq k,\ e_l^T\frac{R^TR_j}{\lambda_j}e_k = e_l^T\frac{R^TR_k}{\lambda_k}e_j,\ j,k,l$ distinct. Now by the standard Cartan trick $-e_k^T\frac{R^TR_l}{\lambda_l}e_j = -e_k^T\frac{R^TR_j}{\lambda_j}e_l = e_l^T\frac{R^TR_j}{\lambda_j}e_k = e_l^T\frac{R^TR_k}{\lambda_k}e_j = -e_j^T\frac{R^TR_k}{\lambda_k}e_l = -e_j^T\frac{R^TR_l}{\lambda_l}e_k = e_k^T\frac{R^TR_l}{\lambda_l}e_j$ for j,k,l distinct, so $e_j^TR^TR_ke_l = 0$ for j,k,l distinct. Keeping account of the prime integral property $\Lambda^T[d\Lambda+\delta R^T(A'V+L^{-1}B)] = 0$ and with $\omega := \sum_{j=1}^n(e_je_j^TR^TR_j\delta+\delta R^TR_je_je_j^T) = -\omega^T$ we have $d\Lambda = \omega\Lambda - \delta R^T(A'V+L^{-1}B)$ and $d\wedge d\Lambda = 0$ becomes the differential system

$$d\wedge\omega - \omega\wedge\omega = -\delta R^TA'R\wedge\delta,\ \omega\wedge\delta - \delta\wedge R^TdR = 0 \Leftrightarrow$$
$$e_j^T[(R^TR_j)_j - (R^TR_k)_k - \sum_l R^TR_le_le_l^TR^TR_l + R^TA'R]e_k = 0,\ j\neq k,$$
$$e_j^TR^TR_le_k = 0 \text{ for } j,k,l \text{ distinct},\ R\subset \mathbf{O}_n(\mathbb{C}) \tag{7}$$

in involution (that is no further conditions appear if one imposes $d\wedge$ conditions and one uses the equations of the system itself) as the compatibility condition for the completely integrable linear system

$$dV = R\delta\Lambda,\ d\Lambda = \omega\Lambda - \delta R^T(A'V+L^{-1}B),\ \Lambda^T\Lambda = -(V^TA'V+2V^TL^{-1}B+|B|^2),$$
$$\omega := \sum_{j=1}^n(e_je_j^TR^TR_j\delta+\delta R^TR_je_je_j^T). \tag{8}$$

The differential system (7) is similar in some aspects to that of Terng's GSGE; in fact for $n = 2,\ A := \mathrm{diag}[a_1^{-1}\quad a_2^{-1}\quad 0]$ and a normalization $a_1^{-1} - a_2^{-1} = 1$ it is the sine-Gordon equation, but the quadratic dependence on R in the first equation for Terng's GSGE is only along the first column Re_1 since a similar phenomenon appears in the definition of ω.

Note that given a solution of (7) (8) will produce an $(n-1)$-dimensional family of deformations $x\subset\mathbb{C}^{2n-1}$ (the prime integral property removes a dimension and translations in u^j another n from the original $2n$-dimensional space of solutions).

One can probably directly analytically develop the B transformation and its BPT of (7), but we shall not insist on it now, since it will appear naturally at the level of the geometric picture from the B transformation of quadrics. Note however that one can deduce at this point that for the first (n,n) entries of A' being a diagonal matrix the 0-soliton $R = I_n$ of (7) will produce Peterson's deformations of quadrics and Λ with some entries zero will produce degenerate solutions (including 0-solitons of paraboloids), so these Peterson's deformations of quadrics will be amenable to explicit computations of their B transforms (at each iteration of the B transformation the $(n-1)$-dimensional family of solutions obtained form the passage of solutions of (7) to solutions of (8) will be heirs of the original $(n-1)$-dimensional family of Peterson's deformations of quadrics), but this is not our interest right now.

4.2. **Deformations of quadrics with center.**

With $V := \sum_{k=1}^n v^ke_k = [v^1\quad ...\quad v^n]^T$ consider the complex unit sphere $X = X(v^1,...,v^n) = \frac{2V+(|V|^2-1)e_{n+1}}{|V|^2+1}$. We have $dX = 2\frac{dV+V^TdV(e_{n+1}-X)}{|V|^2+1}$, so $|dX|^2 = \frac{4|dV|^2}{(|V|^2+1)^2}$.

With $A = A^T\in\mathbf{GL}_{n+1}(\mathbb{C})$ SJ, $H := X^TAX$ we have the QC $x_0 = (\sqrt{A})^{-1}X$ with linear element, unit normal, second fundamental form and Christoffel symbols $|dx_0|^2 = dX^TA^{-1}dX,\ N_0 = \frac{\sqrt{A}X}{\sqrt{H}},\ N_0^Td^2x_0 = -\frac{|dX|^2}{\sqrt{H}},\ \tilde\Gamma_{jk}^l = 0,\ j,k,l$ distinct, $\tilde\Gamma_{jk}^j = -\frac{\partial\log(|V|^2+1)}{\partial v^k},\ \tilde\Gamma_{jj}^k = \frac{\partial\log(\sqrt{H}(|V|^2+1))}{\partial v^k},\ j\neq k,\ \tilde\Gamma_{jj}^j = \frac{\partial\log\frac{\sqrt{H}}{|V|^2+1}}{\partial v^j}$. Note that we have a distinguished tangent vector field $\mathcal{V}_0 := \sum_{k=1}^n\frac{\partial\log\sqrt{H}}{\partial v^k}x_{0v^k} = \frac{4}{(|V|^2+1)^2}(\frac{N_0}{\sqrt{H}} - x_0)$; it has the property $dx_{0v^j} = -\frac{\partial\log(|V|^2+1)}{\partial v^j}dx_0 - d\log(|V|^2+1)x_{0v^j} + (\mathcal{V}_0 + \sum_{k=1}^n\frac{\partial\log(|V|^2+1)}{\partial v^k}x_{0v^k})dv^j - N_0dN_0^Tx_{0v^j}$.

Because $(v^1,...,v^n)$ are isothermic-conjugate and $(u^1,..,u^n)$ are conjugate on x_0, the Jacobian $\frac{\partial(v^1,...,v^n)}{\partial(u^1,...,u^n)}$ has orthogonal columns, so with $\lambda_j := |\frac{\partial V}{\partial u^j}|,\ \Lambda := [\lambda_1\quad ...\quad \lambda_n]^T,\ \delta := \mathrm{diag}[du^1\quad ...\quad du^n]$

we have $dV = R\delta\Lambda,\ R \subset \mathbf{O}_n(\mathbb{C})$. Multiplying the formula for the change of Christoffel symbols $\frac{\partial v^c}{\partial u^l}\Gamma^l_{jk} = \frac{\partial^2 v^c}{\partial u^j \partial u^k} + \frac{\partial v^a}{\partial u^j}\frac{\partial v^b}{\partial u^k}\tilde{\Gamma}^c_{ab}$ on the left with $\frac{\partial v^c}{\partial u^p}$ and summing after c we obtain $\Gamma^p_{jk} = \frac{1}{\lambda_p^2}(\sum_c \frac{\partial v^c}{\partial u^p}\frac{\partial^2 v^c}{\partial u^j \partial u^k} + \sum_c \frac{\partial v^c}{\partial u^p}\frac{\partial v^a}{\partial u^j}\frac{\partial v^b}{\partial u^k}\tilde{\Gamma}^c_{ab}) = \delta_{pk}(\log\frac{\lambda_k}{|V|^2+1})_j + \delta_{pj}(\log\frac{\lambda_j}{|V|^2+1})_k + \delta_{jk}\frac{\lambda_j^2}{\lambda_p^2}(\log\frac{\sqrt{H}(|V|^2+1)}{\lambda_j})_p$, so $\Gamma^j_{jk} = (\log\frac{\lambda_j}{|V|^2+1})_k,\ \Gamma^k_{jj} = \frac{\lambda_j^2}{\lambda_k^2}(\log\frac{\sqrt{H}(|V|^2+1)}{\lambda_j})_k,\ j \neq k,\ \Gamma^j_{jj} = (\log\frac{\lambda_j\sqrt{H}}{|V|^2+1})_j$. We have $h^0_j = \frac{-4\lambda_j^2}{\sqrt{H}(|V|^2+1)^2}$; since $(\log\frac{\lambda_j}{|V|^2+1})_k = \Gamma^j_{jk} = (\log \mathbf{a}_j)_k,\ j \neq k$ we get $\frac{\lambda_j}{|V|^2+1} = \phi_j(u^j)\mathbf{a}_j$; after a change of the u^j variable into itself we can make $\frac{4\lambda_j}{|V|^2+1} = \mathbf{a}_j,\ j = 1, ..., n$, so from (5) $|\Lambda|^2 = -H(|V|^2+1)^2$ and $\Lambda^T d\Lambda = -2dV^T(I_{1,n} + Ve^T_{n+1})A(2V + (|V|^2-1)e_{n+1}) = -2\Lambda^T\delta R^T(I_{1,n} + Ve^T_{n+1})A(2V + (|V|^2-1)e_{n+1})$. Thus again with $\omega := \sum_{j=1}^n (e_je_j^TR^TR_j\delta + \delta R^TR_je_je_j^T) = -\omega^T$ we have $d\Lambda = \omega\Lambda - 2\delta R^T(I_{1,n} + Ve^T_{n+1})A(2V + (|V|^2-1)e_{n+1})$ and imposing $d \wedge d\Lambda = 0$ we obtain the differential system

$$d\wedge\omega - \omega\wedge\omega = -4\delta R^T(I_{1,n} + Ve^T_{n+1})A(I_{1,n} + e_{n+1}V^T)R\wedge\delta,\ \omega\wedge\delta - \delta\wedge R^TdR = 0,$$
$$dV = R\delta\Lambda,\ d\Lambda = \omega\Lambda - 2\delta R^T(I_{1,n} + Ve^T_{n+1})A(2V + (|V|^2-1)e_{n+1}),$$
$$\Lambda^T\Lambda = -(2V + (|V|^2-1)e_{n+1})^TA(2V + (|V|^2-1)e_{n+1}),$$
$$\omega := \sum_{j=1}^n (e_je_j^TR^TR_j\delta + \delta R^TR_je_je_j^T),\ R \subset \mathbf{O}_n(\mathbb{C}) \tag{9}$$

in involution (that is no further conditions appear if one imposes $d\wedge$ conditions and one uses the equations of the system itself).

5. The Bäcklund transformation

5.1. (Isotropic) quadrics without center.

Recall that the Ivory affinity between confocal (I)QWC is given by $x_z = \sqrt{R_z}x_0 + C(z) = \sqrt{R_z}LZ + C(z) = L(I_{1,n}\sqrt{R'_z}Z + e_{n+1}(-I_{1,n}L^{-1}C(z) + e_{n+1})^TZ + L^{-1}C(z)),\ R_z := I_{n+1} - zA,\ C(z) := -(\frac{1}{2}\int_0^z(\sqrt{R_w})^{-1}dw)B, A$ SJ, $\ker(A) = \mathbb{C}e_{n+1},\ B = -e_{n+1}$ for QWC, $\ker(A) = \mathbb{C}f_1,\ B = -\bar{f}_1$ for IQWC. Also $L := (\sqrt{A + e_{n+1}e^T_{n+1}})^{-1}$ for QWC and for IQWC L satisfies $Le_{n+1} = f_1,\ L^T(A + \bar{f}_1\bar{f}_1^T)L = I_{n+1},\ A' := L^TA^2L = I_{1,n}(L^TL)^{-1}I_{1,n}$ SJ.

Consider two points $x^0_0,\ x^1_0 \in x_0$ such that $x^0_0,\ x^1_z$ are in the symmetric TC

$$0 = (N^0_0)^T(x^1_z - x^0_0) \Leftrightarrow\ V_1^T\sqrt{R'_z}V_0 - \frac{|V_1|^2 + |V_0|^2}{2} + (V_0 + V_1)^TL^{-1}C(z) - e^T_{n+1}L^{-1}C(z) = 0 \Leftrightarrow$$
$$|\sqrt{R'_z}V_1 - V_0 + I_{1,n}L^{-1}C(z)|^2 = -zH_1 \Leftrightarrow\ x^1_z = x^0_0 + [x^0_{0v^1_0}\ \ ...\ \ x^0_{0v^n_0}](\sqrt{R'_z}V_1 - V_0 + I_{1,n}L^{-1}C(z)). \tag{10}$$

All these algebraic identities (and other needed later) boil down to those established when a parametrization on (I)QWC was introduced. Thus among the $2n$ functionally independent variables $\{v^j_0, v^j_1\}_{j=1,...,n}$ a quadratic functional relation is established and only $2n-1$ among them remain functionally independent:

$$dV_0^T(\sqrt{R'_z}V_1 - V_0 + I_{1,n}L^{-1}C(z)) = -dV_1^T(\sqrt{R'_z}V_0 - V_1 + I_{1,n}L^{-1}C(z)). \tag{11}$$

Given a deformation $x^0 \subset \mathbb{C}^{2n-1}$ of x^0_0 (that is $|dx^0|^2 = |dx^0_0|^2$) with orthonormal normal frame $N^0 := [N^0_{n+1}\ \ ...\ \ N^0_{2n-1}]$ consider the n-dimensional sub-manifold

$$x^1 = x^0 + [x^0_{v^1_0}\ \ ...\ \ x^0_{v^n_0}](\sqrt{R'_z}V_1 - V_0 + I_{1,n}L^{-1}C(z)) \subset \mathbb{C}^{2n-1} \tag{12}$$

(that is we restrict $\{v^j_1\}_{j=1,...,n}$ to depend only on the functionally independent $\{v^j_0\}_{j=1,...,n}$ and constants in a manner that will subsequently become clear when we shall impose the ACPIA).

Using $dx^0_{v_0^j}=\mathcal{V}^0dv_0^j-N^0(dN^0)^Tx^0_{v_0^j}$ and (11) we have

$$dx^1_z=[-\mathcal{V}^0_0(\sqrt{R'_z}V_0-V_1+I_{1,n}L^{-1}C(z))^T+[x^0_{0v_0^1}\ \dots\ x^0_{0v_0^n}]\sqrt{R'_z}]dV_1-N^0_0(dN^0_0)^T(x^1_z-x^0_0),$$
$$dx^1=[-\mathcal{V}^0(\sqrt{R'_z}V_0-V_1+I_{1,n}L^{-1}C(z))^T+[x^0_{v_0^1}\ \dots\ x^0_{v_0^n}]\sqrt{R'_z}]dV_1-N^0(dN^0)^T(x^1-x^0).$$
(13)

Since our intent is for x^1 to be a B transform (leaf) of the seed x^0 we impose the ACPIA $|dx^1|^2=|dx^1_0|^2$. Keeping in mind $|dx_z|^2=|dx_0|^2-z|dV|^2$ we obtain

$$z|dV_1|^2=|\begin{bmatrix}-i(dN^0_0)^T(x^1_z-x^0_0)\\ -(dN^0)^T(x^1-x^0)\end{bmatrix}|^2.$$

We take advantage now of the conjugate system $(u^1,...,u^n)$ common to x^0_0,x^0 and of the non-degenerate joined second fundamental forms property; according to the principle of symmetry $0\leftrightarrow 1$ we would like $(u^1,...,u^n)$ to be conjugate system to both x^1 and x^1_0 and also that the non-degenerate joined second fundamental forms property holds for x^1_0,x^1.

We augment the column vector $-(dN^0)^T(x^1-x^0)$ with $-i(dN^0_0)^T(x^1_z-x^0_0)$ as the first entry to obtain $\begin{bmatrix}-i(dN^0_0)^T(x^1_z-x^0_0)\\ -(dN^0)^T(x^1-x^0)\end{bmatrix}=\begin{bmatrix}i(N^0_0)^T[dx^0_{0v_0^1}\ \dots\ dx^0_{0v_0^n}]\\ (N^0)^T[dx^0_{v_0^1}\ \dots\ dx^0_{v_0^n}]\end{bmatrix}(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))=$ $\begin{bmatrix}i(N^0_0)^T[dx^0_{0u^1}\ \dots\ dx^0_{0u^n}]\\ (N^0)^T[dx^0_{u^1}\ \dots\ dx^0_{u^n}]\end{bmatrix}\frac{\partial(u^1,...,u^n)}{\partial(v_0^1,...,v_0^n)}(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))=S_0\mathrm{diag}[\lambda_{01}du^1\ \dots\ \lambda_{0n}du^n]$ $\frac{\partial(u^1,...,u^n)}{\partial(v_0^1,...,v_0^n)}(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))=S_0\delta R_0^T(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))$, where $S_0\subset\mathbf{O}_n(\mathbb{C})$ is the orthogonal matrix with columns $\lambda_{0k}^{-1}[ih^0_k\ \ h^{n+1}_k\ \ \dots\ \ h^{2n-1}_k]^T,\ \{h^0_k\}_k,\ \{h^\alpha_k\}_{k,\alpha}$ are the second fundamental forms of x^0_0,x^0 and $\frac{\partial(v_0^1,...,v_0^n)}{\partial(u^1,...,u^n)}=R_0\Lambda_0$. Thus $dV_1=-\frac{1}{\sqrt{z}}M_1S_0\delta R_0^T(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))=R_1\delta\Lambda_1,\ M_1\subset\mathbf{O}_n(\mathbb{C}),\ R_1:=M_1S_0\subset\mathbf{O}_n(\mathbb{C}),\ \Lambda_1:=-\frac{1}{\sqrt{z}}R_0^T(\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z))$. For the prime integral property $|\Lambda_1|^2=-H_1$ we have $|\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z)|^2=-zH_1$. Note that S_0 is undetermined up to multiplication on the left with $R'\subset\mathbf{O}_n(\mathbb{C}),\ R'e_1=e_1$, so the same statement on the right holds for M_1, but M_1S_0 is well defined.

Assuming that $V_1,\ \Lambda_1$ can be found with the properties needed so far, (5) is satisfied and the existence of the B transformation is proved, provided we exhibit an explicit algebraic transformation of the solution $V_0,\ \Lambda_0$ of (8) to the solution $V_1,\ \Lambda_1$ of the same system, with the transformation matrix depending only on the orthogonal R_0,R_1 solutions of (7): this dependence will reveal the analytic B transformation between solutions R_0,R_1 of (7).

We have

$$\sqrt{z}R_1\Lambda_0=\sqrt{R'_z}V_0-V_1+I_{1,n}L^{-1}C(z),$$
$$-\sqrt{z}R_0\Lambda_1=\sqrt{R'_z}V_1-V_0+I_{1,n}L^{-1}C(z) \tag{14}$$

(the symmetry $(0,\sqrt{z})\leftrightarrow(1,-\sqrt{z})$ is required by (11)). From the first relation of (14) we get V_1 as an algebraic expression of $R_1,\ V_0,\ \Lambda_0$; replacing this into the second relation of (14) we get Λ_1 as an algebraic expression of $R_0,\ R_1,\ V_0,\ \Lambda_0$:

$$V_1=\sqrt{R'_z}V_0+I_{1,n}L^{-1}C(z)-\sqrt{z}R_1\Lambda_0,$$
$$\Lambda_1=R_0^T(\sqrt{z}A'V_0+\sqrt{R'_z}R_1\Lambda_0+\sqrt{z}I_{1,n}L^{-1}B). \tag{15}$$

Differentiating the first relation of (15) and using the second we get $R_1\delta R_0^T(\sqrt{z}A'V_0+\sqrt{R'_z}R_1\Lambda_0+\sqrt{z}I_{1,n}L^{-1}B)=\sqrt{R'_z}R_0\delta\Lambda_0-\sqrt{z}[dR_1\Lambda_0+R_1(\omega_0\Lambda_0-\delta R_0^T(A'V_0+L^{-1}B))]$, or the Ricatti equation

$$-dR_1=R_1\omega_0+R_1\delta R_0^TDR_1-DR_0\delta,\ D:=\frac{\sqrt{R'_z}}{\sqrt{z}} \tag{16}$$

in R_1.

Again this is similar to Tenenblat-Terng's B transformation in Terng's interpretation. Imposing the compatibility condition $d\wedge$ on (16) and using the equation itself we thus need: $0=-(R_1\omega_0+R_1\delta R_0^TDR_1-DR_0\delta)\wedge\omega_0+R_1d\wedge\omega_0-(R_1\omega_0+R_1\delta R_0^TDR_1-DR_0\delta)\wedge\delta R_0^TDR_1-R_1\delta\wedge dR_0^TDR_1+R_1\delta R_0^TD\wedge(R_1\omega_0+R_1\delta R_0^TDR_1-DR_0\delta)-DdR_0\wedge\delta=R_1(-\omega_0\wedge\omega_0+d\wedge\omega_0+\delta R_0^TA'R_0\wedge\delta)+DR_0(\delta\wedge\omega_0-R_0^TdR_0\wedge\delta)-R_1(\omega_0\wedge\delta-\delta\wedge R_0^TdR_0)R_0^TDR_1=0$ because R_0 satisfies (7). Therefore (16) is completely integrable and admits solution for any initial value of R_1. If the initial value is orthogonal, we would like the solution to remain orthogonal: $d(R_1R_1^T-I_n)=dR_1R_1^T+R_1dR_1^T=-(R_1\omega_0+R_1\delta R_0^TDR_1-DR_0\delta)R_1^T-R_1(-\omega_0R_1^T+R_1^TDR_0\delta R_1^T-\delta R_0^TD)=-R_1\delta R_0^TD(R_1R_1^T-I_n)-(R_1R_1^T-I_n)DR_0\delta R_1^T$, so $R_1R_1^T-I_n$ is a solution of a linear differential equation and remains 0 if initially it was 0. The fact that R_1 is itself a solution of (7) follows from the symmetry $(0,\sqrt{z})\leftrightarrow(1,-\sqrt{z})$ and the fact that $d\wedge dR_0=0$ (basically we use the converse of the proven results).

Summing up:

Given $R_0\subset\mathbf{O}_n(\mathbb{C})$ solution of (7) and $R_1\subset\mathbf{M}_n(\mathbb{C})$ solution of the Ricatti equation (16), then R_1 remains orthogonal if initially it was orthogonal and in this case it is another solution of (7) (thus producing an $(\frac{n(n-1)}{2}+1)$-dimensional family of solutions). Moreover if $V_0,\ \Lambda_0$ are solutions of (8) associated to R_0 (thus producing a seed deformation $x^0\subset\mathbb{C}^{2n-1}$ of x_0^0), then $V_1,\ \Lambda_1$ given by

$$\begin{bmatrix}V_1\\ \Lambda_1\end{bmatrix}=\sqrt{z}\begin{bmatrix}I_n&0\\0&R_0^T\end{bmatrix}(\begin{bmatrix}D&-I_n\\A'&D\end{bmatrix}\begin{bmatrix}I_n&0\\0&R_1\end{bmatrix}\begin{bmatrix}V_0\\ \Lambda_0\end{bmatrix}+\begin{bmatrix}I_{1,n}\frac{L^{-1}C(z)}{\sqrt{z}}\\ I_{1,n}L^{-1}B\end{bmatrix}),\ D:=\frac{\sqrt{R'_z}}{\sqrt{z}}\tag{17}$$

are solutions of (8) associated to R_1 (thus producing a leaf deformation $x^1\subset\mathbb{C}^{2n-1}$ of x_0^1) and we have the symmetry $(0,\sqrt{z})\leftrightarrow(1,-\sqrt{z})$.

Again one can easily develop the algebraic formula of the BPT at the level of algebraic transformation (17) of solutions (16) and then prove that indeed it induces the BPT of solutions of (7): $\begin{bmatrix}I_n&0\\0&R_2R_1^T\end{bmatrix}(\begin{bmatrix}D_2&-R_3R_0^T\\A'&D_2R_3R_0^T\end{bmatrix}(\begin{bmatrix}D_1&-I_n\\A'&D_1\end{bmatrix}\begin{bmatrix}V_0\\R_1\Lambda_0\end{bmatrix}+\begin{bmatrix}I_{1,n}\frac{L^{-1}C(z_1)}{\sqrt{z_1}}\\I_{1,n}L^{-1}B\end{bmatrix})+\frac{1}{\sqrt{z_1}}\begin{bmatrix}I_{1,n}\frac{L^{-1}C(z_2)}{\sqrt{z_2}}\\I_{1,n}L^{-1}B\end{bmatrix})=\begin{bmatrix}D_1&-R_3R_0^T\\A'&D_1R_3R_0^T\end{bmatrix}(\begin{bmatrix}D_2&-R_2R_1^T\\A'&D_2R_2R_1^T\end{bmatrix}\begin{bmatrix}V_0\\R_1\Lambda_0\end{bmatrix}+\begin{bmatrix}I_{1,n}\frac{L^{-1}C(z_2)}{\sqrt{z_2}}\\I_{1,n}L^{-1}B\end{bmatrix})+\frac{1}{\sqrt{z_2}}\begin{bmatrix}I_{1,n}\frac{L^{-1}C(z_1)}{\sqrt{z_1}}\\I_{1,n}L^{-1}B\end{bmatrix}$, or

$$R_3R_0^T=(D_2-D_1R_2R_1^T)(D_2R_2R_1^T-D_1)^{-1}.\tag{18}$$

The formula thus established the remaining part will follow in a manner similar to Terng's approach. First we need $R_3R_0^T\subset\mathbf{O}_n(\mathbb{C})$, which follows from $(D_2-R_2R_1^TD_1)(D_2R_2R_1^T-D_1)=(R_2R_1^TD_2-D_1)(D_2-D_1R_2R_1^T)$.

Therefore we only need to prove that R_3 given by (18) satisfies (16) for (R_0,z) replaced by $(R_1,z_2),\ (R_2,z_1)$; by symmetry it is enough to prove only one relation. Since $-dR_1=R_1\omega_0+R_1\delta R_0^TD_1R_1-D_1R_0\delta,\ -dR_2=R_2\omega_0+R_2\delta R_0^TD_2R_2-D_2R_0\delta$, we get $d(R_2R_1^T)=-(R_2\omega_0+R_2\delta R_0^TD_2R_2-D_2R_0\delta)R_1^T-R_2(-\omega_0R_1^T+R_1^TD_1R_0\delta R_1^T-\delta R_0^TD_1)=-(R_2R_1^T)R_1\delta R_0^T(D_2R_2R_1^T-D_1)+(D_2-R_2R_1^TD_1)R_0\delta R_1^T$. Thus if we prove the similar relation $d(R_3R_0^T)=-(R_3R_0^T)R_0\delta R_1^T(D_2R_3R_0^T+D_1)+(D_2+R_3R_0^TD_1)R_1\delta R_0^T$, then since $-dR_0=R_0\omega_1-R_0\delta R_1^TD_1R_0+D_1R_1\delta$ we obtain what we want: $-dR_3=R_3\omega_1+R_3\delta R_1^TD_2R_3-D_2R_1\delta$. Differentiating (18) we get $d(R_3R_0^T)(D_2R_2R_1^T-D_1)=-(R_3R_0^TD_2+D_1)d(R_2R_1^T)$; thus we need to prove $[-(R_3R_0^T)R_0\delta R_1^T(D_2R_3R_0^T+D_1)+(D_2+R_3R_0^TD_1)R_1\delta R_0^T](D_2R_2R_1^T-D_1)=-(R_3R_0^TD_2+D_1)[-(R_2R_1^T)R_1\delta R_0^T(D_2R_2R_1^T-D_1)+(D_2-R_2R_1^TD_1)R_0\delta R_1^T]$. The terms containing $R_1\delta R_0^T$ become $D_2+R_3R_0^TD_1=(R_3R_0^TD_2+D_1)R_2R_1^T$ which follows directly from (18) and the terms containing $R_0\delta R_1^T$ become $R_0\delta R_1^T(D_2R_3R_0^T+D_1)(D_2R_2R_1^T-D_1)=(R_3R_0^T)^T(R_3R_0^TD_2+D_1)(D_2-R_2R_1^TD_1)R_0\delta R_1^T$ which follows from $(D_2R_3R_0^T+D_1)(D_2R_2R_1^T-D_1)=D_2^2-D_1^2=(\frac{1}{z_2}-\frac{1}{z_1})I_n=(R_3R_0^T)^T(R_3R_0^TD_2+D_1)(D_2-R_2R_1^TD_1)$.

To prove the existence of the $\mathcal{M}_p$ configuration we need only prove the existence of the $\mathcal{M}_3$ configuration (discrete deformations in $\mathbb{C}^{2n-1}$ of x_0 will be obtained by considering $\mathbb{Z}^n$ lattices of B

transformations which are subsets of infinite Möbius configurations, similarly to Bobenko-Pinkall's approach [7]).

Consider $(D_1D_2)^{-1}[(D_1^2-D_2^2)D_1R_1(D_1R_1-D_2R_2)^{-1}-D_1^2]=R_3R_0^{-1}=(D_1D_2)^{-1}[(D_1^2-D_2^2)D_2R_2(D_1R_1-D_2R_2)^{-1}-D_2^2]$, $R_5R_0^{-1}=(D_1D_3)^{-1}[(D_3^2-D_1^2)D_1R_1(D_3R_4-D_1R_1)^{-1}-D_1^2]$, $R_6R_0^{-1}=(D_2D_3)^{-1}[(D_2^2-D_3^2)D_2R_2(D_2R_2-D_3R_4)^{-1}-D_2^2]$; thus with $\square:=(D_2^2-D_3^2)D_1R_1+(D_3^2-D_1^2)D_2R_2+(D_1^2-D_2^2)D_3R_4$ we have $(D_2R_3R_0^{-1}-D_3R_5R_0^{-1})^{-1}R_1=[(D_1^2-D_2^2)(D_1R_1-D_2R_2)^{-1}-(D_3^2-D_1^2)(D_3R_4-D_1R_1)^{-1}]^{-1}=(D_1R_1-D_2R_2)\square^{-1}(D_3R_4-D_1R_1)$ and similarly $(D_3R_6R_0^{-1}-D_1R_3R_0^{-1})^{-1}R_2=(D_1R_1-D_2R_2)\square^{-1}(D_2R_2-D_3R_4)$. Now $D_1[(D_2^2-D_3^2)D_2R_3R_0^{-1}(D_2R_3R_0^{-1}-D_3R_5R_0^{-1})^{-1}R_1-D_2^2R_1]=(D_1^2D_2R_2-D_2^2D_1R_1)\square^{-1}(D_2^2-D_3^2)(D_3R_4-D_1R_1)-D_2^2D_1R_1=(D_1^2D_2R_2-D_2^2D_1R_1)\square^{-1}(D_3^2-D_1^2)(D_2R_2-D_3R_4)-D_1^2D_2R_2=D_2[(D_3^2-D_1^2)D_1R_3R_0^{-1}(D_3R_6R_0^{-1}-D_1R_3R_0^{-1})^{-1}R_2-D_1^2R_2]$, so the very lhs and rhs provide the good definition of and afford themselves the name $D_1D_2D_3R_7$.

We shall now show that the facets dx^1, rigidly transported by rolling to facets centered at x_z^1 when x^0 rolls on x_0^0, will cut the tangent spaces of x_z^1 along rulings.

In this case it is useful to extend the orthonormal normal frame N^0 with a 0 column vector on the left and call the extended 'orthonormal normal frame' thus obtained still orthonormal frame; all computations will remain valid as long as one uses transpose instead of inverse. The facet dx^1 will be transported to the facet $dx_z^1+N_0^0(dN_0^0)^T(x_z^1-x_0^0)+\sqrt{z}[0\ N_0^0\quad e_{n+2}\ \dots\ e_{2n-1}]MdV_1=([x^1_{zv_1^1}\ \dots\ x^1_{zv_1^n}]+\frac{N_0^0}{\sqrt{H_0}}(\sqrt{R_z'}V_0-V_1+I_{1,n}L^{-1}C(z))^T+\sqrt{z}[0\ N_0^0\quad e_{n+2}\ \dots\ e_{2n-1}]M)dV_1$, $dV_1=\frac{1}{\sqrt{z}}R_1\delta R_0^T(\sqrt{R_z'}V_1-V_0+I_{1,n}L^{-1}C(z))$, where $M\subset\mathbf{O}_n(\mathbb{C})$ is determined up to multiplication on the left with an orthogonal matrix which fixes e_1 (due to the indeterminacy of the rolling in the normal bundle) by $M^Te_1=\pm i\frac{\sqrt{R_z'}V_0-V_1+I_{1,n}L^{-1}C(z)}{\sqrt{zH_0}}$ (because of the ACPIA requirement $z(e_1^TMdV_1)^2=-[(dN_0^0)^T(x_z^1-x_0^0)]^2=-\frac{1}{H_0}((\sqrt{R_z'}V_0-V_1+I_{1,n}L^{-1}C(z))^TdV_1)^2$). The vectors of the facet are obtained by replacing δ with diagonal constant matrices Δ; let $\Delta':=\frac{1}{\sqrt{z}}R_1\Delta R_0^T(\sqrt{R_z'}V_1-V_0+I_{1,n}L^{-1}C(z))$. Thus we need $e_j^TM\Delta'=0,\ j=3,...,n$, so $M\Delta'=e_1^TM\Delta'e_1+e_2^TM\Delta'e_2$; from $(N_z^1)^Tw=0$ we obtain $e_2^TM\Delta'=\pm ie_1^TM\Delta'$; after scaling we obtain $w=\pm[x^1_{zv_1^1}\ \dots\ x^1_{zv_1^n}](e_1\pm iM^Te_2)$. Now the condition $w^TAR_z^{-1}w=0$ that w is a ruling on x_z^1 becomes $|e_1\pm iM^Te_2|^2=0$ and it is thus satisfied.

Now the notion of W congruence for $n=2$ requires that the asymptotic directions on the two focal surfaces correspond, so the natural generalization for $n\ge 3$ is that the asymptotic directions on the two focal sub-manifolds x^0,x^1 should correspond. An asymptotic direction $v=a^jx_j$ must satisfy $a^ja^kh^\alpha_{jk}=0,\ \alpha=n+1,...,2n-1$; in our case (restricting to conjugate system) we need $\sum_j(a^j)^2\lambda_j\frac{h_j^\alpha}{\lambda_j}=0$; keeping account of the orthogonal properties of the non-degenerate joined second fundamental forms we get $a^j=\pm1$ and thus they are the same for x^0,x^1.

5.2. **Quadrics with center.**

Recall that we have the Ivory affinity $x_z=\sqrt{R_z}(\sqrt{A})^{-1}X,\ X=\frac{2V+(|V|^2-1)e_{n+1}}{|V|^2+1}$. Note

$$\frac{(|V|^2+1)^2}{4}\sum_{k=1}^n x_{zv^k}x_{zv^k}^T+x_zx_z^T=A^{-1}-zI_{n+1}.$$

Consider two points $x_0^0,\ x_0^1\in x_0$ such that $x_0^0,\ x_z^1$ are in the symmetric TC

$$(x_z^1-x_0^0)^TN_0^0=0\Leftrightarrow X_1^T\sqrt{R_z}X_0=1\Leftrightarrow x_z^1=x_0^0+\frac{(|V_0|^2+1)^2}{4}\sum_{k=1}^nX_1^T\sqrt{R_z}X_{0v_0^k}x^0_{0v_0^k}\Leftrightarrow$$

$$(2V_0+(|V_0|^2-1)e_{n+1})^T\sqrt{R_z}(2V_1+(|V_1|^2-1)e_{n+1})=(|V_0|^2+1)(|V_1|^2+1). \tag{19}$$

Thus among the $2n$ functionally independent variables $\{v_0^j, v_1^j\}_{j=1,...,n}$ a quadratic functional relation is established and only $2n-1$ among them remain functionally independent:

$$X_1^T\sqrt{R_z}dX_0 = -X_0^T\sqrt{R_z}dX_1. \tag{20}$$

Given a deformation $x^0 \subset \mathbb{C}^{2n-1}$ of x_0^0 (that is $|dx^0|^2 = |dx_0^0|^2$) with orthonormal normal frame $N^0 := [N_{n+1}^0 \ \ ... \ \ N_{2n-1}^0]$ consider the n-dimensional sub-manifold

$$x^1 = x^0 + \frac{(|V_0|^2+1)^2}{4}\sum_{k=1}^n X_1^T\sqrt{R_z}X_{0v_0^k}x^0_{v_0^k} \subset \mathbb{C}^{2n-1} \tag{21}$$

(that is we restrict $\{v_1^j\}_{j=1,...,n}$ to depend only on the functionally independent $\{v_0^j\}_{j=1,...,n}$ and constants in a manner that will subsequently become clear when we shall impose the ACPIA).

Using $dx^0_{v_0^k} = -\frac{\partial\log(|V_0|^2+1)}{\partial v_0^k}dx^0 - d\log(|V_0|^2+1)x^0_{v_0^k} + (\mathcal{V}^0 + \sum_{j=1}^n\frac{\partial\log(|V_0|^2+1)}{\partial v_0^j}x^0_{v_0^j})dv_0^k - N^0(dN^0)^Tx^0_{v_0^k}$ and (20) we have

$$dx_z^1 = (-\mathcal{V}_0^0X_0^T + \frac{(|V_0|^2+1)^2}{4}\sum_{k=1}^n x^0_{0v_0^k}X^T_{0v_0^k})\sqrt{R_z}dX_1 - N_0^0(dN_0^0)^T(x_z^1 - x_0^0),$$

$$dx^1 = (-\mathcal{V}^0X_0^T + \frac{(|V_0|^2+1)^2}{4}\sum_{k=1}^n x^0_{v_0^k}X^T_{0v_0^k})\sqrt{R_z}dX_1 - N^0(dN^0)^T(x^1 - x^0). \tag{22}$$

Since our intent is for x^1 to be a B transform (leaf) of the seed x^0 we impose the ACPIA $|dx^1|^2 = |dx_0^1|^2$. Keeping in mind $|dx_z|^2 = |dx_0|^2 - z|dX|^2$ we obtain

$$z|dX_1|^2 = |\begin{bmatrix}-i(dN_0^0)^T(x_z^1 - x_0^0)\\ -(dN^0)^T(x^1 - x^0)\end{bmatrix}|^2.$$

We take advantage now of the conjugate system $(u^1, ..., u^n)$ common to x_0^0, x^0 and of the non-degenerate joined second fundamental forms property; according to the principle of symmetry $0 \leftrightarrow 1$ we would like $(u^1, ..., u^n)$ to be conjugate system to both x^1 and x_0^1 and also that the non-degenerate joined second fundamental forms property holds for x_0^1, x^1. We obtain by computations similar to the (I)QWC case $dV_1 = R_1\delta\Lambda_1,\ \Lambda_1 := \frac{(|V_1|^2+1)(|V_0|^2+1)}{2\sqrt{z}}R_0^T[X_1^T\sqrt{R_z}X_{0v_0^1} \ \ ... \ \ X_1^T\sqrt{R_z}X_{0v_0^n}]^T$ (note that the prime integral property $|\Lambda_1|^2 = -H_1(|V_1|^2+1)^2$ is satisfied).

Thus

$$\sqrt{z}R_0\Lambda_1 = (I_{1,n} + V_0e_{n+1}^T)\sqrt{R_z}(2V_1 + (|V_1|^2-1)e_{n+1}) - V_0(|V_1|^2+1),$$

$$-\sqrt{z}R_1\Lambda_0 = (I_{1,n} + V_1e_{n+1}^T)\sqrt{R_z}(2V_0 + (|V_0|^2-1)e_{n+1}) - V_1(|V_0|^2+1) \tag{23}$$

(the symmetry $(0, \sqrt{z}) \leftrightarrow (1, -\sqrt{z})$ follows from (20)). From the second relation of (23) we get V_1 as an algebraic expression of $R_1,\ V_0,\ \Lambda_0$; replacing this into the first relation of (23) we get Λ_1 as an algebraic expression of $R_0,\ R_1,\ V_0,\ \Lambda_0$:

$$V_1 = -\frac{\sqrt{z}R_1\Lambda_0 + I_{1,n}\sqrt{R_z}(2V_0 + (|V_0|^2-1)e_{n+1})}{e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2-1)e_{n+1}) - |V_0|^2 - 1},$$

$$\Lambda_1 = 2R_0^T\frac{(I_{1,n} + V_0e_{n+1}^T)[\sqrt{z}A(2V_0 + (|V_0|^2-1)e_{n+1}) - \sqrt{R_z}(I_{1,n} + e_{n+1}V_1^T)R_1\Lambda_0] + V_0V_1^TR_1\Lambda_0}{e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2-1)e_{n+1}) - |V_0|^2 - 1}. \tag{24}$$

Differentiating the second relation of (23) we get $-\sqrt{z}dR_1\Lambda_0 - \sqrt{z}R_1[\omega_0\Lambda_0 - 2\delta R_0^T(I_{1,n} + V_0e_{n+1}^T)A(2V_0 + (|V_0|^2-1)e_{n+1})] - 2I_{1,n}\sqrt{R_z}(I_{1,n} + e_{n+1}V_0^T)R_0\delta\Lambda_0 = [e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2-1)e_{n+1}) - |V_0|^2 -$

$1]R_1\delta\Lambda_1 + 2V_1[e_{n+1}^T\sqrt{R_z}(I_{1,n} + e_{n+1}V_0^T) - V_0^T]R_0\delta\Lambda_0$, or equivalently the Ricatti equation

$$\begin{aligned}-dR_1 = {} & R_1\omega_0 + 2I_{1,n}\frac{\sqrt{R_z}}{\sqrt{z}}(I_{1,n} + e_{n+1}V_0^T)R_0\delta - 2R_1\delta R_0^T(I_{1,n} + V_0e_{n+1}^T)\frac{\sqrt{R_z}}{\sqrt{z}}R_1 \\ & +2R_1\delta R_0^T\frac{[(I_{1,n} + V_0e_{n+1}^T)\sqrt{R_z}e_{n+1} - V_0][\Lambda_0^T + (2V_0^T + (|V_0|^2 - 1)e_{n+1}^T)\frac{\sqrt{R_z}}{\sqrt{z}}R_1]}{e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2 - 1)e_{n+1}) - |V_0|^2 - 1} \\ & -2\frac{[R_1\Lambda_0 + I_{1,n}\frac{\sqrt{R_z}}{\sqrt{z}}(2V_0 + (|V_0|^2 - 1)e_{n+1})][e_{n+1}^T\sqrt{R_z}(I_{1,n} + e_{n+1}V_0^T) - V_0^T]}{e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2 - 1)e_{n+1}) - |V_0|^2 - 1}R_0\delta\end{aligned} \tag{25}$$

in R_1.

Note that with $M := I_{1,n}\frac{\sqrt{R_z}}{\sqrt{z}}(I_{1,n} + e_{n+1}V_0^T),\ N := I_{1,n}\frac{\sqrt{R_z}}{\sqrt{z}}(2V_0 + (|V_0|^2 - 1)e_{n+1})$, $W := (I_{1,n} + V_0e_{n+1}^T)\sqrt{R_z}e_{n+1} - V_0,\ U := e_{n+1}^T\sqrt{R_z}(2V_0 + (|V_0|^2 - 1)e_{n+1}) - |V_0|^2 - 1$ we have $dN = 2MdV_0,\ dU = 2W^TdV_0,\ dM = CdV_0^T,\ dW = cdV_0$ and (25) can be written as

$$-dR_1 = R_1\omega_0 + 2MR_0\delta - 2R_1\delta R_0^TM^TR_1 + \frac{2}{U}R_1\delta R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2}{U}(R_1\Lambda_0 + N)W^TR_0\delta.$$

Imposing the $d\wedge$ condition on (25) and using the equation itself we need $0 = -[R_1\omega_0 + 2MR_0\delta - 2R_1\delta R_0^TM^TR_1 + \frac{2}{U}R_1\delta R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2}{U}(R_1\Lambda_0 + N)W^TR_0\delta]\wedge[\omega_0 - 2\delta R_0^TM^TR_1 + \frac{2}{U}\delta R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2}{U}\Lambda_0W^TR_0\delta] - R_1\delta R_0^T(2M^T - \frac{2}{U}WN^T)\wedge[R_1\omega_0 + 2MR_0\delta - 2R_1\delta R_0^TM^TR_1 + \frac{2}{U}R_1\delta R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2}{U}(R_1\Lambda_0 + N)W^TR_0\delta] + R_1d\wedge\omega_0 + 2CdV_0^TR_0\wedge\delta + 2MdR_0\wedge\delta - 2R_1\delta\wedge R_0^TdR_0R_0^TM^TR_1 + 2R_1\delta\wedge R_0^TdV_0C^TR_1 + \frac{2}{U}R_1\delta\wedge R_0^TdR_0R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2c}{U}R_1\delta\wedge R_0^TdV_0(\Lambda_0^T + N^TR_1) - \frac{2}{U}R_1\delta\wedge R_0^TW(d\Lambda_0^T + 2dV_0^TM^TR_1) + \frac{4}{U^2}R_1\delta\wedge R_0^TW(\Lambda_0^T + N^TR_1)W^TdV_0 - \frac{2}{U}(R_1d\Lambda_0 + 2MdV_0)W^TR_0\wedge\delta - \frac{2}{U}(R_1\Lambda_0 + N)cdV_0^TR_0\wedge\delta + \frac{4}{U^2}(R_1\Lambda_0 + N)W^TW^TdV_0R_0\wedge\delta - \frac{2}{U}(R_1\Lambda_0 + N)W^TdR_0\wedge\delta = \frac{4}{U^2}[W^T(dV_0 - R_0\delta\Lambda_0)\wedge[(R_1\Lambda_0 + N)W^TR_0\delta - R_1\delta R_0^TW(\Lambda_0^T + N^TR_1)] + (|\Lambda_0|^2 - |N|^2)R_1\delta R_0^TWW^TR_0\wedge\delta] + \frac{2}{U}[-cR_1\delta R_0^T\wedge(dV_0 - R_0\delta\Lambda_0)(\Lambda_0^T + N^TR_1) - c(R_1\Lambda_0 + N)(dV_0 - R_0\delta\Lambda_0)^TR_0\wedge\delta - R_1\delta\wedge R_0^TW[\Lambda_0^T\omega_0 + d\Lambda_0^T + 2(dV_0 - R_0\delta\Lambda_0)^TM^TR_1] + [R_1(\omega_0\Lambda_0 - d\Lambda_0) + 2M(R_0\delta\Lambda_0 - dV_0)]W^TR_0\wedge\delta + (R_1\Lambda_0 + N)W^TR_0(\delta\wedge\omega_0 - R_0^TdR_0\wedge\delta) - R_1(\omega_0\wedge\delta - \delta\wedge R_0^TdR_0)R_0^TW(\Lambda_0^T + N^TR_1) + 2R_1\delta R_0^T(WN^TM + M^TNW^T)R_0\wedge\delta] - 2MR_0(\delta\wedge\omega_0 - R_0^TdR_0\wedge\omega_0) + R_1[-\omega_0\wedge\omega_0 + d\wedge\omega_0 - 4\delta R_0^TM^TMR_0\wedge\delta + (\omega_0\wedge\delta - \delta\wedge R_0^TdR_0)R_0^TM^TR_1] + 2C(dV_0 - R_0\delta\Lambda_0)^TR_0\wedge\delta + 2R_1\delta\wedge R_0^T(dV_0 - R_0\delta\Lambda_0)C^TR_1$. Using the fact that $V_0,\ \Lambda_0,\ R_0$ are solutions of (9) we need $0 = R_1\delta R_0^T[|\Lambda_0|^2 - |N|^2]WW^T + UW[(2V_0 + (|V_0|^2 - 1)e_{n+1})^TA(I_{1,n} + e_{n+1}V_0^T) + N^TM] + U[(I_{1,n} + V_0e_{n+1}^T)A(2V_0 + (|V_0|^2 - 1)e_{n+1}) + M^TN]W^T - U^2[M^TM + (I_{1,n} + V_0e_{n+1}^T)A(I_{1,n} + e_{n+1}V_0^T)]R_0\wedge\delta$; using $|\Lambda_0|^2 - |N|^2 = -\frac{1}{z}[(|V_0|^2 + 1)^2 - (U + |V_0|^2 + 1)^2],\ (2V_0 + (|V_0|^2 - 1)e_{n+1})^TA(I_{1,n} + e_{n+1}V_0^T) + N^TM = -\frac{1}{z}[U(W + V_0)^T + (|V_0|^2 + 1)W^T],\ M^TM + (I_{1,n} + V_0e_{n+1}^T)A(I_{1,n} + e_{n+1}V_0^T = \frac{1}{z}[I_n - WW^T - WV_0^T - V_0W^T]$ this is straightforward. Therefore (25) is completely integrable and admits solution for any initial value of R_1. If the initial value is orthogonal, we would like the solution to remain orthogonal: $d(R_1R_1^T - I_n) = dR_1R_1^T + R_1dR_1^T = -[R_1\omega_0 + 2MR_0\delta - 2R_1\delta R_0^TM^TR_1 + \frac{2}{U}R_1\delta R_0^TW(\Lambda_0^T + N^TR_1) - \frac{2}{U}(R_1\Lambda_0 + N)W^TR_0\delta]R_1^T - R_1[-\omega_0R_1^T + 2\delta R_0^TM^T - 2R_1^TMR_0\delta R_1^T + \frac{2}{U}(\Lambda_0 + R_1^TN)W^TR_0\delta R_1^T - \frac{2}{U}\delta R_0^TW(\Lambda_0^TR_1^T + N^T)] = 2(R_1R_1^T - I_n)MR_0\delta R_1^T + 2R_1\delta R_0^TM^T(R_1R_1^T - I_n) - \frac{2}{U}R_1\delta R_0^TWN^T(R_1R_1^T - I_n) - \frac{2}{U}(R_1R_1^T - I_n)NW^TR_0\delta R_1^T$, so $R_1R_1^T - I_n$ is a solution of a linear differential equation and remains 0 if initially it was 0. The fact that R_1 is itself a solution of (9) follows from the symmetry $(0, \sqrt{z}) \leftrightarrow (1, -\sqrt{z})$ and the fact that $d\wedge dR_0 = 0$ (basically we use the converse of the proven results).

We shall now show that the facets dx^1, rigidly transported by rolling to facets centered at x_z^1 when x^0 rolls on x_0^0, will cut the tangent spaces of x_z^1 along rulings.

In this case it is useful to extend the orthonormal normal frame N^0 with a 0 column vector on the left and call the extended 'orthonormal normal frame' thus obtained still orthonormal frame; all computations will remain valid as long as one uses transpose instead of inverse. The facet dx^1 will be transported to the facet $dx_z^1 + N_0^0(dN_0^0)^T(x_z^1 - x_0^0) + 2\sqrt{z}[0\ N_0^0\ \ e_{n+2}\ \ \dots\ \ e_{2n-1}]\frac{MdV_1}{|V_1|^2+1} =$

$([x^1_{zv^1_1}\ \ \dots\ \ x^1_{zv^n_1}]+\frac{N^0_0}{\sqrt{H_0}}X_0^T\sqrt{R_z}[X_{1v^1_1}\ \ \dots\ \ X_{1v^n_1}]+2\sqrt{z}[0\ N^0_0\ \ e_{n+2}\ \ \dots\ \ e_{2n-1}]\frac{M}{|V_1|^2+1})dV_1,\ dV_1=\frac{(|V_1|^2+1)(|V_0|^2+1)}{2\sqrt{z}}R_1\delta R_0^T(X_1^T\sqrt{R_z}[X_{0v^1_0}\ \ \dots\ \ X_{0v^n_0}])^T$, where $M\subset \mathbf{O}_n(\mathbb{C})$ is determined up to multiplication on the left with an orthogonal matrix which fixes e_1 (due to the indeterminacy of the rolling in the normal bundle) by $M^Te_1=\pm i\frac{(|V_1|^2+1)(X_0^T\sqrt{R_z}[X_{1v^1_1}\ \ \dots\ \ X_{1v^n_1}])^T}{2\sqrt{zH_0}}$ (because of the ACPIA requirement $\frac{4z(e_1^TMdV_1)^2}{(|V_1|^2+1)^2}=-[(dN^0_0)^T(x^1_z-x^0_0)]^2=-\frac{1}{H_0}(X_0^T\sqrt{R_z}[X_{1v^1_1}\ \ \dots\ \ X_{1v^n_1}]dV_1)^2)$. The vectors of the facet are obtained by replacing δ with diagonal constant matrices Δ; let $\Delta':=\frac{(|V_1|^2+1)(|V_0|^2+1)}{2\sqrt{z}}R_1\Delta R_0^T(X_1^T\sqrt{R_z}[X_{0v^1_0}\ \ \dots\ \ X_{0v^n_0}])^T$. Thus we need $e_j^TM\Delta'=0,\ j=3,...,n$, so $M\Delta'=e_1^TM\Delta'e_1+e_2^TM\Delta'e_2$; from $(N^1_z)^Tw=0$ we obtain $e_2^TM\Delta'=\pm ie_1^TM\Delta'$; after scaling we obtain $w=\pm[x^1_{zv^1_1}\ \ \dots\ \ x^1_{zv^n_1}](e_1\pm iM^Te_2)$. Now the condition $w^TAR_z^{-1}w=0$ that w is a ruling on x^1_z becomes $|e_1\pm iM^Te_2|^2=0$ and it is thus satisfied.

The W congruence property follows in a manner similar to the (I)QWC case.

6. The Bianchi Theorems on confocal quadrics

6.1. **The Ivory affinity provides a rigid motion.**

6.2. **The second iteration of the tangency configuration.**

6.3. **Möbius configurations.**

6.4. **Homographies and confocal quadrics.**

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